Geometric rigidity for epipelagic automorphic data on symplectic groups

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Abstract

We propose a method to establish geometric rigidity of automorphic data, and work it out for the case of epipelagic data in symplectic groups. These notes are part of Zhiwei Yun's project group in Arizona Winter School 2022 part of which was the author.

1 Introduction

Let k be a finite field, and X a smooth projective geometrically connected curve X over k. Let $F = k(X)$ be the field of rational functions on X. F_x is the completion of the local ring at $x \in X$, and $\mathcal{O}_x$ is the ring of integers of F_x. $\mathbb{A}_F$ is the ring of adeles, and then $\prod G(\mathcal{O}_x)$ is the maximal compact open subgroup of $G(\mathbb{A}_F)$. Let $S \subset |X|$ be finite. Automorphic data (K_S, χ_S) we call a collection of compact open subgroups K_x for each $G(F_x)$ and finite order characters χ_x of K_x.

Automorphic forms are certain functions on $G(F)\backslash G(\mathbb{A})$ transforming in a way prescribed by χ_x under the right action of K_x. Using a generalized version of Weil's observation, we can see these as functions on $Bun_G(K_S)$, the moduli stack of G-torsors equipped with extra structure coming from the automorphic data.

In [Yun14], Zhiwei Yun defines geometrically rigid automorphic data, a property of automorphic data that forces the corresponding space of automorphic forms to be essentially one-dimensional. Under the function-sheaf correspondence, this gives a Hecke eigensheaf whose eigenvalue gives rise to interesting local systems. A point $\mathcal{E} \in Bun_G(K_S)$ is called relevant if some canonical maps $Aut(\mathcal{E}) \to k$ have to be zero, otherwise it is called irrelevant. Essentially, this has to happen so that an automorphic form can possibly have some nonzero value on the double coset $[g]$. The automorphic data (K_S, χ_S) are called geometrically rigid if they give rise to only a single relevant point in each connected component of $Bun_G(K_S)$.

In these notes, we will try to verify the rigidity of epipelagic data in $G = Sp_{2n}$ by carefully constructing concretely these maps. Notice that since $Bun_G(K_S)$ is connected, there must be a unique relevant point.

More specifically, the argument tries to establish that for all but one point, which will generally be the most generic one, the automorphism group will be "large enough" that the evaluation map cannot be trivial. In particular, one finds a specific subgroup of $Bun_G(K_S)$ that could not possibly map to 0.

2 Setup

Let $G = Sp_{2n}$, $X = \mathbb{P}^1$ and $S = \{0, \infty\}$ with level subgroups

- $K_0 = P_0^{opp}$, the inverse image of an opposite Siegel parabolic under the reduction to k map.
- $K_\infty = P_\infty^+$, the kernel of the reduction map to a Siegel parabolic.

Then $Bun_G(K_S)$ has the following geometric interpretation: It parametrizes quantiples $\mathscr{E} = (V, \omega, L_0, L_\infty, \{l_1, \dots, l_n\})$ of a vector bundle of rank $2n$ with a symplectic form (V, ω), equipped with Lagrangian submanifolds L_0, L_∞ of the fibers at $\{0, \infty\}$, together with a basis of L_∞.

3 Geometric rigidity

1 The evaluation map

Let $\mathscr{E} \in Bun_G(K_S)$. Since $X = \mathbb{P}^1$, we can use Birkhoff's decomposition for vector bundles, together with the symplectic structure to write

$$V = \left(\oplus_{i=1}^n \mathscr{O}(\lambda_i)\right) \oplus \left(\oplus_{i=1}^n \mathscr{O}(-\lambda_i)\right)$$

for a multiset of integers $\lambda_i \in \mathbb{Z}$. Since L_∞ is Lagrangian, it can only be contained in one of the corresponding $\mathscr{O}(\pm\lambda_i)$, which we can assume to be the plus one since we do not assume λ_i to be positive. Notice that the same property holds for L_0 but after fixing the choice of $\pm\lambda_i$ we cannot have any information on the corresponding signs for L_0.

Yun's map now would be a composition

$$Aut(\mathscr{E}) \to Sym^2(L_\infty) \oplus Sym^2(L_\infty^\vee) \to Sym^2(St) \oplus Sym^2(St^\vee) \to k \to \overline{\mathbb{Q}_p^\times}. \quad (1)$$

Let $\phi \in Aut(\mathscr{E})$. We can write ϕ globally as a $2n \times 2n$ matrix of global sections of $Hom(\mathscr{O}(a), \mathscr{O}(b))$ for the corresponding choices of a, b. Notice that by tensor-Hom adjunction,

$$Hom(\mathscr{O}(a), \mathscr{O}(b)) \cong \mathscr{O}(a) \otimes \mathscr{O}(b)^\vee \cong \mathscr{O}(a) \otimes \mathscr{O}(-b) \cong \mathscr{O}(a-b),$$

and also that these global sections have conditions at fibers on $\{0, \infty\}$ imposed by the preservation of the extra structure.

Now, the first step in 1 is given by taking evaluation of ϕ at ∞. Notice that this gives a matrix with $n \times n$ blocks, satisfying the following conditions.

1. The diagonal $n \times n$ blocks should be the identity by preservation of the basis of L_∞.
2. Entries corresponding to $a - b < 0$ should be zero because $\mathcal{O}(a-b)$ has no global sections.
3. In the lower diagonal block B the entries as global sections should be symmetric by the symplectic structure and evaluate at 0 at infinity by preservation of L_∞. Therefore, they are divisible by the uniformizer at ∞ which we call τ.
4. The upper diagonal block A is symmetric by the symplectic structure.

The first step of the map 1 now is sending ϕ to the evaluations at ∞ of $(A, B/\tau)$, and we get the next step by using the basis to write the pair $(A, B/\tau)$ as symmetric matrices. Then we choose a *stable* vector $(U,V) \in Sym^2(St) \oplus Sym^2(St^\vee)$ - more on this later - and we get the third step of the map by the trace pairing

$$(A, B/\tau) \to Tr(AV) + Tr((B/\tau)U)$$

and then the last step is some fixed character.

2 The stability condition

In this section we work out what stability means. It is written for completeness and can be skipped, as it is enough to know the statement of the lemma below.

A point $(U,V) \in Sym^2(St) \oplus Sym^2(St^\vee)$ is called stable, if under the GL_n action given by $g(U,V) = (gU^t g, {}^t g^{-1} V g^{-1})$ it has closed orbit and finite stabilizer.

Lemma 1. *A vector (U,V) is stable if and only if UV is non-singular and has distinct eigenvalues.*

Proof. □

3 The case of a trivial bundle

If $\mathcal{E}$ is the trivial bundle, we can identify L_0, L_∞ as living in the same vector space. Then there is a unique up to automorphisms point $\mathcal{E}_0$ where $L_0 \cap L_\infty = \{0\}$. It turns out $\mathcal{E}_0$ is the unique relevant point of $Bun_G(K_S)$. Indeed, by the considerations in the evaluation map section we can see that ϕ is comprised of global sections of $\mathcal{O}$, ie. constant functions, which by the vanishing at 0 and ∞ correspondingly means that ϕ basically has to be the identity matrix. Since $Aut(\mathcal{E})$ is just a point, it gives a trivial map to k and therefore $\mathcal{E}_0$ is relevant.

Now assume $\mathcal{E}_1$ is a point for which $L_0 \cap L_\infty = l$ for some line l. Other cases are even easier. By a similar argument we see that B/τ has to be zero, and A also has to be zero everywhere except the first row and last column, which also have to be symmetric but otherwise are free. Npw if we assume the evaluation map is trivial for any choice of A, this means that $Tr(AV) = 0$ for any A.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & 0 & \cdots & 0 \end{pmatrix}, V = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & & & \\ \vdots & & V' & \\ v_{n1} & & & \end{pmatrix}.$$

We have

$$\mathrm{Tr}(AV) = a_{11}v_{11} + a_{12}v_{21} + \cdots + a_{1n}v_{n1}$$

Since we can choose the a_{1i} freely, that means one complete row of V is zero, which contradicts V being a part of a stable pair by lemma 1. Therefore, $\mathscr{E}_1$ has to be irrelevant. The same argument works for all other points corresponding to the trivial bundle.

4 The case of a non-trivial bundle

In the case of a nontrivial bundle we will try to find a subgroup that cannot map to zero by showing there are enough global sections in specific parts of $\phi \in Aut(\mathscr{E})$. For all but finitely many of them we can show there exists a free row/column like in the previous section.

Indeed, notice that for $|a - b| \geq 2$, an entry corresponding to $\mathscr{O}(|a - b|)$ can be chosen freely since we have at most two conditions and $dim(\mathscr{O}(|a - b|)) \geq 3$. But this excludes for example ie. that the largest λ_i has distance more than 1 from any other. Even with just this argument, one can work the remaining cases for $G = Sp_4$ by hand.

More careful arguments can exclude a lot more cases eg. even with the difference being 0, if there is no condition coming from S the entry can be still chosen freely, and if there is no condition from at least one point the it is enough that the difference has absolute value 1. Also one can modify the argument to consider other subgroups rather than one free row/column.

4 Conclusion

Even to complete the argument for epipelagic data in symplectic groups one seems to need at least one more idea. One advantage of this method though is that it reduces rigidity to some kind of elaborate linear algebra problem.

References

[Yun14] Zhiwei Yun. Rigidity in automorphic representations and local systems. *arXiv preprint arXiv:1405.3035*, 2014.