Note on Iwahori-Hecke algebras

Kostas I. Psaromiligkos

Abstract

The purpose of these notes is to review the structure theory of Iwahori-Hecke algebras, and provide some geometric analogs of certain objects involved. For the first one we will follow closely the beautiful expository paper [HP09], and for the second one we will draw ideas and statements from [Zhu16].

1 Introduction

In this section, we are going to set up notation and motivate the study of Iwahori-Hecke algebras.

The setting is the following. Let G be a reductive group over a non-archimedean local field F with valuation ring $\mathcal{O}$ and maximal ideal $p = (\pi)$ where π is a uniformizer. Let $k = \mathcal{O}/p$ be the residue field. Also let $B = AN$ a Borel with torus A and unipotent radical N. Then $I = im^{-1}(G(\mathcal{O}) \to G(k))(B(k))$ is an Iwahori subgroup. Fix a Haar measure dg. We consider the Iwahori-Hecke algebra $H = C_c(I\backslash G/I)$ with multiplication being convolution with respect to the Haar measure.

Readers with not a lot of familiarity on reductive groups can think of a reductive group as a group with similar structural properties to $GL_2(F)$, which we will review below and use it as our working example.

For $GL_2(F)$ the Borel subgroup consists of upper triangular matrices, the torus of the diagonal matrices, the unipotent radical of upper triangular matrices with ones on the diagonal, and the Iwahori will be elements in

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix}$$

The reason for studying this setting is the following. There is an equivalence of categories between smooth admissible representations of G and non-degenerate modules of the convolution algebra of locally constant compactly supported distributions on G, called the Hecke algebra and denoted by $\mathcal{H}$. This algebra is non-unital, but has a lot of idempotents which give a filtration of this Hecke algebra by compact open subgroups of G. Specifically, for a compact open subgroup K let e_K be the characteristic distribution supported in K giving it measure 1. Then obviously $e_K^2 = e_K$. We define $H_K = e_K \mathcal{H} e_K$. This algebra has e_K as a unit, and by a standard exercise a module M of $\mathcal{H}$ gives a (possibly trivial) module $e_K M$ of H_K.

The equivalence of categories mentioned above restricts to an equivalence of the category of smooth admissible representations generated by K-fixed vectors and H_K-modules.

The smaller the subgroup, the more representations it is going to see. We will study only the cases where K is a maximal compact or Iwahori subgroup.

2 Basic structure

Let $M = C_c(A_{\mathcal{O}} N \backslash G / I)$ to be the ***universal unramified principal series module.*** M is obviously an H-module by the action given by convolution.

M will be very important to us because of the lemma.

Lemma 1. *M is a free module of rank 1. In particular, $H \cong End_H(M)$, the isomorphism given by $h \to v_1 h$.*

Proof. We will show that the map described in the theorem, in terms of the bases T_w, v_w is upper triangular with non-zero diagonal.

Indeed, this follows from the fact that $NxI \cap IyI \neq \varnothing \implies x \leq y$ in the Bruhat ordering. □

Consider $R = X_*(A)$ to be the group algebra of the cocharacter lattice, and W the Weyl group. Also let $H_0 = C_c(I \backslash K / I)$ be the finite Hecke algebra.

R, H_0 are naturally subalgebras of H. We have

Lemma 2. *The map $R \otimes H_0 \to H$ given by convolution is an isomorphism of vector spaces.*

3 Bernstein presentation

The Bernstein presentation lemma is given by the following proposition.

Proposition 3. *H is the algebra generated over R by the elements T_w subject to the relations*

1. $T_{s_a}^2 = (q-1)T_{s_a} + qT_e$.
2. $T_{s_a}\pi^\mu = \pi^{s_a(\mu)}T_{s_a} + (q-1)\frac{\pi^\mu - \pi^{s_a(\mu)}}{1-\pi^{-a^\vee}}$.

The quadratic relation is given by just combinatorial considerations on double cosets, which is how we generally compute this kind of integrals. For the intertwining relation though we will need the so-called intertwining operators which are discussed in the next section.

For now notice that while the intertwining relation maybe appear to not be an equation in H_I, the related fractional term is actually an element of R because it is equal to a finite sum.

Nonetheless, it will be useful to us to be able to do algebraic manipulations so we let L be the fraction field of R, $H_I^L = L \otimes_R H_I$, and $M^L = L \otimes_R M$.

We still have that M^L is a free rank one module with the same generator, and $H_I^L = End_{H_I^L}(M^L)$.

Now that we have expanded our algebra and actually have denominators, one can note that the intertwining relation for H_I is equivalent to the following relation in H_I^L.

$$\left(T_{s_a} - (q-1)\frac{1}{1-\pi^{-a^\vee}}\right)\pi^\mu = \pi^{s_a(\mu)}\left(T_{s_a} - (q-1)\frac{1}{1-\pi^{-a^\vee}}\right).$$

We are therefore looking for an endomorphism M^L that satisfies the commutation relation $I_a\pi^\mu = \pi^{s_a(\mu)}I_a$.

We will see this in the next section.

4 Intertwiners

To show the intertwining relation, we will introduce operators I_a which have the property that $I_a \circ \pi^\mu = \pi^{s_a(\mu)} \circ I_a$, and then identify these with elements in $H_I^L = L \otimes_R H_I$ (notice that these elements will have denominators in R). This will be done by computing the action of I_a on the spherical vector.

Consider a a simple root, $B_a = s_a B s_a^{-1}$ and N_a the unipotent radical of B_a. $U_a = (N_a \cap N)(N_a \cap \overline{N})$ is one dimensional.

Then we have the operator

$I'_a : C_c(A(\mathcal{O})N\backslash G/I) \to C(A(\mathcal{O})N_a\backslash G/I)$ defined by

$$I'_a(\phi)(g) = \int_{U_a} \phi(u_a g)du_a.$$

Notice that the integral is well-defined due to the compact support, but does not necessarily give a function with compact support. I'_a is obviously R-linear.

We define $I_a : C_c(A(\mathcal{O})N\backslash G) \to C(A(\mathcal{O})N\backslash G)$ by $I_a(\phi)(g) = I'_a(\phi)(s_a g)$ where s_a is a representative in G. Now I_a has the property we want, as I_a by definition is (R, s_a) semi-linear. That is, $I_a \circ \pi^\mu = \pi^{s_a(\mu)} \circ I_a$.

5 Normalized intertwiners and center

5.1 Normalized intertwiners

The intertwiners I_a are not elements of R because of the denominators $1 - \pi^{-a^\vee}$. So it makes sense to define $J_a = (1 - \pi^{-a^\vee})I_a$. Indeed, J_a carries M to M and is an H-module homomorphism, therefore $J_a \in H_I$. We also have the relations $J_a\pi^\mu = \pi^{s_a(\mu)}J_a$ and $J_a^2 = (1 - q^{-1}\pi^{-a^\vee})(1 - q^{-1}\pi^{a^\vee})$.

Now the elements J_w have all the properties we wanted, but their multiplication has still the deficit that $J_{w_1w_2} = J_{w_1}J_{w_2}$ only if $l(w_1w_2) = l(w_1) + l(w_2)$. This has to do with the quadratic relation. We can fix this problem and attain even more canonical intertwiners by defining the *normalized intertwiners*

$$K_w = \left(\prod_{a\in R_w} \frac{1}{1-q\pi^{-a^\vee}}\right) \cdot J_w.$$

Indeed, now $K_{s_a}^2 = 1$ and thus $K_{w_1w_2} = K_{w_1}K_{w_2}$ for ***all*** $w_1, w_2 \in W$.

Notice that $K_w \in H_I^L$ but $K_w \notin H_I$.

5.2 Center

As we have mentioned, one of the nice properties of H_I we will now explore and that greatly simplifies its representation theory is that it has a large center. We begin with the following lemma that follows easily from what we have done.

Lemma 4. *The W-invariants of R are contained in the center, ie. $R^W \subseteq Z(H_I)$.*

Proof. Any element of R commutes with every other, so it is enough to prove commutation with the elements T_{s_a}. But indeed, by the commutation relation $J_a \pi^\mu = \pi^{s_a(\mu)} J_a$ we have that for any $r \in R^W$, r commutes with $(1 - \pi^{-a^\vee}) T_{s_a}$. That by a simple calculation means that the bracket of r, T_{s_a} is annihilated by $(1 - \pi^{-a^\vee})$, and since H_I is a free R-module the bracket vanishes. □

It will turn out that we actually have $Z(H_I) = R^W$.

Lemma 5. $R^W = Z(H_I)$.

Proof. By the relations satisfied by the normalized intertwiners, we have a group homomorphism $W \to (H_I^L)^\times$ given by $w \to K_w$. This extends to a homomorphism from the twisted group algebra $f : L\langle W \rangle \to H_I^L$.

But now $L\langle W \rangle$ is a matrix algebra over L^W, therefore it is simple and f is injective. But now both algebras have dimension $|W|^2$ over L^W, thus f is an isomorphism.

The center of a matrix algebra are the scalar diagonal matrices, so $Z(H_I^L) = L^W$. Intersecting with R, we get $Z(H_I) = R^W$. □

6 Classical Satake transform

The Satake transform gives a complete description of the spherical Hecke algebra H_K. It turns out that $H_K \cong R^W$.

Indeed, M^K is a free rank one module over H_K, and we have $H_K \cong End_{H_K}(M^K)$. M^K is also a free rank 1 R-module, and therefore for each $h \in H_K$ there is a unique element $h^\vee \in R$ such that

$$h \star m = m \cdot h^\vee.$$

$h^\vee$ is called the ***Satake transform*** of h. If we take m to be the spherical vector and act by the normalized intertwiners, which preserve the spherical vector by the Gindikin-Karpelevich formula, we find that actually $h^\vee \in R^W$.

Theorem 6 (Satake transform). *The Satake transform is an isomorphism.*

Proof. As previously, we write the change of basis matrix between the two algebras according to properly chosen bases and use an upper triangular argument.

We choose the bases $h_\mu = 1_{K\pi^\mu K}$ where μ ranges over dominant cocharacters for H_K, and $s_\nu = \sum_{\lambda \in W\nu} \pi^\lambda$ for R^W.

Then the change of basis matrix has entries

$$c_{\mu\nu} = \delta_B(\pi^\nu)^{-1/2} \int_N 1_{K\pi^\mu K}(n\pi^\nu) dn$$

This coefficient is non-negative and non-zero if and only if $K\pi^\mu K$ meets $N\pi^\nu$. So $c_{\mu\mu}$ is obviously nonzero and a careful analysis performed by Bruhat and Tits shows that $c_{\mu\nu} = 0$ unless $\nu \le \mu$. □

7 Geometrization

Some of the objects defined above have geometric counterparts. We will now explore this by considering the geometric Satake transform, and try to find counterparts of each object.

The geometric Satake transform is a categorification/geometrization of the classical one. We first need some notation.

Let Gr_G be the affine Grassmannian of G and L^+G be the positive loop group. Then the geometric analog of H_K is the category $P_{L^+G}(Gr_G)$ of L^+G-equivariant perverse sheaves over Gr_G.

Indeed, sending a perverse sheaf to the euler characteristic of its hypercohomology gives a map from $P_{L^+G}(Gr_G)$ to H_K, by the identification of H_K with K-binvariant functions.

One equips this category with a convolution by taking fusion products of perverse sheaves.

The geometric analog of R^W is the representation ring of the Langlands dual group $\hat{G}$. The reason for that is that characters of said representations are invariant functions under conjugation, and we compose this observation with the identification $t//W \cong \mathfrak{g}//G$. The torus of the Langlands dual will indeed give the group algebra of the cocharacter lattice of G.

Theorem 7 (Geometric Satake transform). *There is an equivalence of categories*

$$P_{L^+G}(Gr_G) \cong R(\hat{G}),$$

that sends a fusion product of perverse sheaves to the tensor product of the corresponding representations.

We will not provide a proof of this theorem in these notes, but only mention that the idea of the proof is that one applies the Tannakian formalism to the category $P_{L^+G}(Gr_G)$ under the symmetric monoidal functor of hypercohomology, to identify $P_{L^+G}(Gr_G)$ with the representation category of an algebraic group $\tilde{G}$. Using the explicit construction of that group supplied by the formalism, we prove that the root datum of $\tilde{G}$ is dual to the one of G. By the combinatorial classification of algebraic groups via their root data, we get $\tilde{G} \cong \hat{G}$.

Remark 8. *The geometric Satake transform implies the classical one.*

Indeed, fix for technical reasons an isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$. Taking K-groups and tensoring by $\overline{\mathbb{Q}}_l$, the geometric Satake equivalence gives

$$K(P_{L^+G}(Gr_G)) \otimes \overline{\mathbb{Q}}_l \cong K(R(\hat{G})) \otimes \overline{\mathbb{Q}}_l.$$

Then we use ι to get an equivalence

$$K(P_{L^+G}(Gr_G)) \otimes \mathbb{C} \cong K(R(\hat{G})) \otimes \mathbb{C}.$$

The right hand side is identified naturally with elements of R^W by taking characters. The left hand side is identified with the spherical Hecke algebra via the function-sheaf dictionary, as described at the start of this section.

References

[HP09] Kottwitz Haines and Prasad. Iwahori-Hecke Algebras, 2009.

[Zhu16] Xinwen Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence, 2016.