THE UNIVERSITY OF CHICAGO

THE LAFFORGUE VARIETY AND TOPICS IN GEOMETRIC REPRESENTATION THEORY

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
KONSTANTINOS PSAROMILIGKOS

CHICAGO, ILLINOIS
AUGUST 2023

As you set out for Ithaka hope your road is a long one, full of adventure, full of discovery.

Constantine Cavafy, *Ithaca*

TABLE OF CONTENTS

ACKNOWLEDGMENTS

My profound gratitude goes to my advisor, Châu Ngô and my secondary advisor Sasha Beilinson, for their continued support in becoming a mathematician. I am extremely happy to have had the chance to work with them, and they greatly helped me to shape my way of thinking about mathematics.

I warmly thank Anne-Marie Aubert, Sasha Beilinson, Micah Gay, Mathilde Gerbelli-Gauthier, Aristides Kontogeorgis, Zhilin Luo, Siddharth Mahendraker, Luca Migliorini, Benedict Morrissey, Châu Ngô, Takeshi Saito, Yiannis Sakellaridis, Will Sawin, Maarten Solleveld, Minh-Tâm Trinh, Griffin Wang, Yujie Xu and Tong Zhou for their interest and many helpful discussions, without which this thesis would not have been possible. I am particularly indebted to Anne-Marie Aubert, Sasha Beilinson, Micah Gay, Mathilde Gerbelli-Gauthier, Zhilin Luo, Siddharth Mahendraker, Benedict Morrissey, Châu Ngô, Maarten Solleveld and Tong Zhou for valuable feedback on parts of or even complete drafts of this thesis.

During this work, except the University of Chicago, I was also supported by a Graduate Research Fellowship from the Onassis foundation.

Life during a PhD can be stressful, especially with the interception of COVID-19, so I also have to thank the many friends who supported me in hard times. From mathematicians in the University of Chicago not previously mentioned, I am deeply grateful to Faidon Andriopoulos, Jason Kountouridis, Nikiforos Mimikos-Stamatopoulos, Santiago Chaves-Aguilar, Mark Cerenzia and Pallav Goyal. From my friends in Greece where I spent the pandemic, an incomplete but special list of people whom I cannot thank enough includes Constance, Alexandra, Nikos, Myronas, Nikolas, Vassilis, Aspasia, Stelios, and Marilia.

Finally, my warmest thanks go to my family for helping me and guiding me during my whole life.

ABSTRACT

This thesis consists of two separate projects.

In the first project, we construct the Lafforgue variety, an affine scheme parametrizing the simple modules of a non-commutative algebra R over any field k, provided that the center $Z(R)$ is finitely generated and R is finite as a $Z(R)$-module. Applying our construction in the case of Hecke algebras of Bernstein components, we derive a characterization for the irreducibility of induced representations in terms of the vanishing of a generalized discriminant on the Bernstein variety. We explicitly compute the discriminant in the case of an Iwahori-Hecke algebra of a split reductive p-adic group.

We additionally give potential applications to the Local Langlands conjecture via comparison of Hecke algebras on the group and Galois sides, as in the ABPS conjectures. In particular, we construct a Bernstein variety for the Galois side of the Local Langlands correspondence and conjecture that the Lafforgue varieties of the two sides are isomorphic.

In the second project, we prove that character sheaves have nilpotent singular support in any characteristic, partially extending the work of [MV88] and [Gin89] to positive characteristic. We do this by introducing a category of tame perverse sheaves and studying its functorial properties.

CHAPTER 1

THE LAFFORGUE VARIETY

1.1 Introduction

1.1.1 Summary

The category of smooth representations $\mathcal{M}(G)$ of a reductive p-adic group G is not semisimple. Instead, there is a splitting of $\mathcal{M}(G)$ as a direct product of indecomposable categories by virtue of the Bernstein decomposition theorem

$$\mathcal{M}(G) \cong \prod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathcal{M}_{\mathfrak{s}}(G),$$

see Theorem 1.2.20 or [BD84, Proposition 2.10], indexed by the set of connected components $\mathfrak{B}(G)$ of the Bernstein variety $\Omega(G)$.

In particular, to each smooth irreducible representation $\pi \in \mathrm{Irr}(G)$ we can attach uniquely up to G-conjugation its cuspidal support $\mathbf{sc}(\pi) \coloneqq (M, \sigma)_G$, where M is a Levi subgroup of G and σ a supercuspidal irreducible representation of M such that π embeds in the parabolic induction $i_M^G(\sigma)$. Then, $\Omega(G)$ parametrizes the set of conjugation classes of possible cuspidal supports. As $i_M^G(\sigma)$ is of finite length, $\Omega(G)$ is also a finite-to-one parametrizing space for $\mathrm{Irr}(G)$. We define $\mathrm{Irr}^{\mathfrak{s}}(G) \subseteq \mathrm{Irr}(G)$ to be the subset of irreducible representations with cuspidal support in $\mathfrak{s}$. The splitting provided by Bernstein's decomposition theorem induces the partition

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G) \tag{1.1}$$

on the level of irreducible objects.

With the intent of studying $\mathcal{M}(G)$, various versions of Hecke algebras have been introduced, with the property that their module category is equivalent to some subcategory of

$\mathcal{M}(G)$. The centers of Hecke algebras are isomorphic to subrings of the ring of regular functions of $\Omega(G)$. The study of the representation theory of p-adic reductive groups via Hecke algebras has provided an abundance of beautiful results, see for example [IM65], [KL87], [AMS22].

In [Laf16], Laurent Lafforgue predicted the existence of an algebraic variety classifying smooth irreducible representations of a p-adic reductive group with regular functions being generated by traces over Hecke algebras.

More precisely, Lafforgue's proposed construction works as follows. For every smooth representation (V,π) of G we have $V = \bigcup_K V^K$ where K ranges over all compact open subgroups $K \leq G$ and V^K denotes the subspace of K-fixed vectors. As a consequence of Bernstein's admissibility theorem, if V is also irreducible, V^K is finite. Let $\mathcal{M}_K(G)$ be the full subcategory of smooth representations generated by K-fixed vectors and $\mathcal{H}_K(G)$ the Hecke algebra of K-biinvariant locally constant compactly supported distributions on G - see subsection 2.3. We have

$$\mathcal{M}_K(G) \cong \mathcal{M}(\mathcal{H}_K(G))$$

where $\mathcal{M}(\mathcal{H}_K(G))$ denotes the module category of $\mathcal{H}_K(G)$. Therefore, $\mathrm{Irr}_K(G) \cong \mathrm{Irr}(\mathcal{H}_K(G))$.

We consider for each $r \in \mathcal{H}_K(G)$ a function $f_r : \mathrm{Irr}(\mathcal{H}_K(G)) \to \mathbb{C}$ defined by $f_r(V) = \mathrm{tr}_V(r)$. Let $\mathcal{T}_K$ be the ring of functions on $\mathrm{Irr}(\mathcal{H}_K(G))$ generated by all $f_r, r \in \mathcal{H}_K(G)$, which we call the *ring of traces*. The set $\mathrm{Irr}(\mathcal{H}_K(G))$ can be naturally embedded in $\mathrm{Spec}(\mathcal{T}_K)$, since any $V \in \mathrm{Irr}(\mathcal{H}_K(G))$ gives a geometric point via the evaluation homomorphism. Lafforgue predicted that $\mathrm{Irr}_K(G) \cong \mathrm{Irr}(\mathcal{H}_K(G))$ embeds as an open dense subscheme of $\mathrm{Spec}(\mathcal{T}_K)$.

We prove the existence of this basic object in the following more general setting: Let R be a not necessarily commutative k-algebra over any field k, such that the center $Z(R)$ is finitely generated and R is finite as a $Z(R)$-module. Let A be any subalgebra of $Z(R)$ such that R is a finite A-module. An irreducible R-module will be finite as a consequence of Schur's lemma and the finiteness assumption. Therefore, by the same procedure we can define the ring of traces T_R of R over A. We define $\mathrm{Laf}_{R/A} := \mathrm{Spec}(T_R)$ to be the *Lafforgue variety.*

The subalgebra $A \subseteq Z(R)$, acts by a character on any simple module, and thus we get a natural projection $\mathrm{Laf}_{R/A} \to \mathrm{Spec}(A)$.

We can now state our main theorem, where we also remove any assumptions on k. Notice that our definition of the ring T_R will be more involved in the case $\mathrm{char}(k) > 0$ - see subsection 3.3.

Theorem 1.1.1. $\mathrm{Irr}(R)$ *forms the set of* $\bar{k}$*-points of a dense Zariski open subscheme* $\mathrm{iLaf}_{R/A}$ *of* $\mathrm{Laf}_{R/A}$. *The projection* $p : \mathrm{Laf}_{R/A} \to \mathrm{Spec}(A)$ *is finite.*

The main difficulty in proving Theorem 1.1.1 is that Lafforgue's proposed construction of the trace ring T_R does not provide much information on its structure. What we thus need is a framework in algebraic geometry which gives rise to a more workable definition of the trace ring. This framework turns out to be a non-commutative generalization of the classical Hilbert-Chow morphism.

Our strategy is as follows: we will first construct a non-commutative Hilbert scheme for the finite A-algebra R, which is a proper A-scheme Q. Next, using the trace (or the determinant in the positive characteristic case), we construct a morphism from the Hilbert scheme to an affine A-scheme V. The morphism $Q \to V$ can then be shown to factor through a closed subscheme of V which is finite over $\mathrm{Spec}(A)$, of which the coordinate ring can be identified with T_R in the case $\mathrm{char}(k) = 0$.

We first apply our results to the question of irreducibility for a parabolically induced representation from a cuspidal datum, a long studied subject, see for example [BZ76], [Mul79], [Kat81], [KL87]. In particular, we consider the Hecke algebra $\mathcal{H}_{\mathfrak{s}}(G)$ of a Bernstein component $\mathfrak{s} \in \mathfrak{B}(G)$, with the property that its module category $\mathcal{M}(\mathcal{H}_{\mathfrak{s}}(G))$ satisfies

$$\mathcal{M}_{\mathfrak{s}}(G) \cong \mathcal{M}(\mathcal{H}_{\mathfrak{s}}(G)).$$

In [Sol22], it was shown that $\mathcal{H}_{\mathfrak{s}}$ is almost Morita equivalent, ie. the categories of finite-dimensional modules are equivalent, to a twisted affine Hecke algebra $H_{\mathfrak{s}}$.

Applying Theorem 1.1.1 to the case of the Hecke algebra $\mathcal{H}_{\mathfrak{s}}$ of a Bernstein component $\mathfrak{s} \in \mathfrak{B}(G)$ and its center $Z_{\mathfrak{s}}$, we get a finite map

$$p : \mathrm{Laf}_{\mathcal{H}_{\mathfrak{s}}/Z_{\mathfrak{s}}} \to \Omega_{\mathfrak{s}}$$

from the Lafforgue variety to the Bernstein variety. When restricted to $\mathrm{iLaf}_{\mathcal{H}_{\mathfrak{s}}/Z_{\mathfrak{s}}}$, it agrees with Bernstein's cuspidal support map upon identifying $\mathrm{Irr}^{\mathfrak{s}}(G)$ with $\mathrm{iLaf}_{\mathcal{H}_{\mathfrak{s}}/Z_{\mathfrak{s}}}(\mathbb{C})$.

Returning to the general case of any algebra R satisfying our condition and a central subalgebra A, we stratify $\mathrm{Spec}(A)$ according to the cardinality of the fibers of p. If $R = \mathcal{H}_{\mathfrak{s}}$, $A = Z_{\mathfrak{s}}$, the parabolic induction $i_M^G(\sigma)$ from a cuspidal datum $(M, \sigma)_G \in \mathrm{Spec}(Z_{\mathfrak{s}})$ is irreducible if and only if $|p^{-1}(M, \sigma)| = 1$, ie. on the open dense stratum X_0 of the cardinality stratification.

If A is regular over a field k of characteristic 0 and R is also a locally free A-module, we have another concrete description of the stratification. Fixing a central character $\chi : A \to k$, a simple R-module with central character χ corresponds to an $R_\chi = \left(R \otimes_{A,\chi} k\right)$-module. We can then describe the stratification by studying the rank of the Jacobson radical of R_χ.

We mainly apply this result to the open dense stratum X_0. We define a notion of a *generalized discriminant* $d_{R/A}$, which is a principal ideal of A, with the property that the complement of its zero set in $\mathrm{Spec} A$ is X_0.

Induced representations are irreducible for generic cuspidal data, so they are irreducible exactly on X_0. We choose a regular central subalgebra $A \subseteq Z_{\mathfrak{s}}$ with $f : \mathrm{Spec}(Z_{\mathfrak{s}}) \to \mathrm{Spec}(A)$ finite. Using the Jacobson stratification previously described, we prove the following.

Theorem 1.1.2. *Let $(M, \sigma)_G$ be a cuspidal datum. Then, $i_M^G(\sigma)$ is irreducible if and only* *if* $(M,\sigma) \in X_0$. *Outside of the singular locus* $Z(f)$, *this is equivalent to*

$$d_{\mathcal{H}_{\mathfrak{s}}/A}(f(M,\sigma)) \neq 0.$$

We develop computational methods for generalized discriminants that work well with explicit presentations as in Solleveld's theorem [Sol22]. In particular, we explicitly calculate the discriminant for the unramified Bernstein component and the Iwahori-Hecke algebra of a split reductive p-adic group to retrieve results about irreducibility of principal series appearing, for example, in [Kat81].

Another approach in parametrizing smooth representations of reductive groups is the (enhanced) Local Langlands Correspondence. It asserts a natural bijection of $\mathrm{Irr}(G)$ in the group side with a set $\Phi_e(G)$ of conjugacy classes of enhanced Langlands parameters in the Galois side. In particular, there is a partition

$$\Phi_e(G) = \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi_e^{\mathfrak{s}^\vee}(G),$$

corresponding to (1.1) in the Galois side [AMS18]. Aubert, Moussaoui and Solleveld also constructed a twisted affine Hecke algebra $H_{\mathfrak{s}^\vee}$ such that $\Phi_e^{\mathfrak{s}^\vee}(G) \cong \mathrm{Irr}(H_{\mathfrak{s}^\vee})$ as sets [AMS21]. They also proved many cases of the enhanced Local Langlands Correspondence by comparing the Hecke algebras on the group side and the Galois side [AMS22]. It is believed that the two Hecke algebras are always almost Morita equivalent, which implies the bijection. Building on their work, we consider the weaker geometric Conjecture 1.6.4, which asserts the existence of a commutative diagram

$$\begin{array}{ccc} \mathrm{Laf}_{H_{\mathfrak{s}}/Z_{\mathfrak{s}}} & \xrightarrow{\cong} & \mathrm{Laf}_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(Z_{\mathfrak{s}}) & \xrightarrow{\cong} & \mathrm{Spec}(Z_{\mathfrak{s}^\vee}) \end{array}$$

We believe Conjecture 1.6.4 may help clarify proofs of known cases and is easier to establish than the full almost Morita equivalence. With the purpose of studying the Conjecture, we lay out a Bernstein-type theory on the Galois side. In particular, we prove that $\Omega_{\mathfrak{s}^\vee} := \mathrm{Spec}(Z_{\mathfrak{s}^\vee})$ parametrizes the set of equivalence classes $\mathfrak{s}^\vee$, and the finite projection

$$\mathrm{Laf}_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}} \to \Omega_{\mathfrak{s}^\vee}$$

when restricted to $\mathrm{iLaf}_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}}(\mathbb{C})$ agrees with the cuspidal support map on the Galois side.

1.1.2 Outline

In Section 2, we recall classical results in the structure theory and the classification of p-adic reductive groups and the construction of the Bernstein variety. We also recall various versions of Hecke algebras, both to amend possible confusion stemming from the existence of multiple algebras going by that name in the literature and to make our treatment more self-contained.

In Section 3, we define our non-commutative generalizations of the Hilbert scheme and the Hilbert-Chow morphism. We construct the trace map and carry out the proof of Theorem 1.1.1 in the characteristic zero case. Then, we construct the determinant map to treat the positive characteristic case. We also show the Lafforgue variety construction is independent of the auxiliary choice of a central subalgebra $A \subseteq Z(R)$.

In Section 4, we construct the Jacobson stratification. We use it to define a notion of equivalence of algebras based on them having isomorphic Lafforgue varieties, and to derive a geometric bijection on the open dense subschemes corresponding to the irreducible representations in such a case.

In Section 5, we define generalized discriminants and study the case of Hecke algebras to prove Theorem 1.1.2. We prove properties of discriminants to make them more amenable to calculation, including a generalization of the classical behavior of the discriminant in a tower of extensions of number rings to general commutative algebras, which doesn't seem to appear in the literature for our case, using the generalized Riemann-Hurwitz formula. In particular, we show the following.

Lemma 1.1.3. *For a tower of extensions $C/B/A$ such that A, B are commutative and regular, C is commutative, C is free of rank n as a B-module and B is free as an A-module, we have that*

$$d_{C/A} = (d_{B/A})^n \cdot N_{B/A}(d_{C/B}),$$

where $N_{B/A}$ is the norm function.

As an application, we compute the discriminant for the case of an Iwahori-Hecke algebra of a split reductive p-adic group.

In Section 6, we recall relevant material to the enhanced Local Langlands correspondence. We recall results of [AMS21] and [AMS22]. We state Conjecture 1.6.4, and lay out the Bernstein theory for the Galois side of the correspondence. We show the Conjecture implies a weak version of enhanced Local Langlands. We also show it follows from conjecturally true statements and in many cases known results.

1.2 The Bernstein variety and Hecke algebras

We recall the classical construction of the Bernstein variety and state the Bernstein decomposition theorem [BD84]. We also recall various versions of Hecke algebras used to study $\mathcal{M}(G)$ or $\mathcal{M}_{\mathfrak{s}}(G)$. Following [HKP09], we illuminate the structure of the Iwahori-Hecke algebra using intertwiners. We summarize recent progress in describing $\mathcal{M}_{\mathfrak{s}}(G)$ via (twisted) affine Hecke algebras [AMS21], [AMS22].

1.2.1 Reductive groups and their classification

Let $\mathbf{G}$ be a reductive algebraic group defined over a non-archimedean local field F with ring of integers $\mathcal{O}$ and uniformizer τ. We denote by $k = \mathcal{O}/\tau\mathcal{O}$ the residue field and $q = |k|$. Let $G = \mathbf{G}(F)$ be the group of F-points. Let F_{sep} be a separable closure of F.

A subgroup $\mathbf{P}$ of $\mathbf{G}$ is called *parabolic* if $\mathbf{G}/\mathbf{P}$ is compact. Then, $\mathbf{P} = \mathbf{MU}$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{P}$ and $\mathbf{M}$ is a *Levi subgroup* of $\mathbf{G}$. If $\mathbf{P}$ is also solvable, it is called a *Borel subgroup.*

Definition 1.2.1. *G is called quasi-split over F if $\mathbf{G}$ has a Borel subgroup $\mathbf{B}$ defined over F.*

An algebraic group $\mathbf{T}$ is called a *torus* if over F_{sep} it is isomorphic to $\mathbb{G}_m^r$. The F-rank of $\mathbf{T}$ is the largest integer s such that there is an embedding $\mathbb{G}_m^s \hookrightarrow \mathbf{T}$ defined over F. By definition, the F_{sep}-rank of $\mathbf{T}$ is r. If the F-rank of $\mathbf{T}$ is equal to its F_{sep}-rank, we call $\mathbf{T}$ *split* over F.

Definition 1.2.2. *$\mathbf{G}$ is called split over F, or simply G is split, if there exists a maximal torus $\mathbf{T} \subseteq \mathbf{G}$ that is split over F.*

Remark 1.2.3. *If G is split, then G is also quasi-split. In that case, Borel subgroups are the minimal parabolic subgroups, and parabolic subgroups are exactly the subgroups containing a Borel. Over F_{sep}, every reductive group $\mathbf{G}$ is split.*

Split reductive groups have a simple characterization. We need the following definition.

Definition 1.2.4. *A root datum is a quadruple $\mathcal{R} = (X^*, \Phi, X_*, \Phi^\vee)$ such that*

- *X^*, X_* are free abelian groups of finite rank equipped with a perfect pairing $(\cdot, \cdot)$ with values in $\mathbb{Z}$.*

- *$\Phi, \Phi^\vee$ are finite subsets of X^*, X_* equipped with a bijection $\vee$ such that $(a, a^\vee) = 2$ for all $a \in \Phi$.*

- *For each $a \in \Phi$ the map $s_a : X^* \to X^*$ defined by $s_a(x) := x - (x, a^\vee)a$ induces an automorphism of the root datum. We also require the symmetric condition for $\Phi^\vee$.*

If for any $a \in \Phi$ we have that $2a \notin \Phi$, $\mathcal{R}$ is called reduced.

Let $\mathbf{T}$ be a maximal split over F torus of $\mathbf{G}$.

Proposition 1.2.5. *Let $X^*(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m)$ be the character lattice of $\mathbf{T}$, $X_*(\mathbf{T}) := \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$ be the cocharacter lattice, $\Phi(\mathbf{G}, \mathbf{T}) \subseteq X^*(\mathbf{T})$ the weights of the induced $\mathbf{T}$-action on the Lie algebra $\mathfrak{g}$, and $\Phi^\vee(G, \mathbf{T}) \subseteq X_*(\mathbf{T})$ the coweights. For any $a \in \Phi(\mathbf{G}, \mathbf{T})$, there is a unique $a^\vee \in \Phi^\vee(\mathbf{G}, \mathbf{T})$ such that $a(a^\vee(x)) = x^2$ for any $x \in \mathbb{G}_m$. Then,*

$$\mathcal{R} = (X^*(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T}), X_*(\mathbf{T}), \Phi^\vee(\mathbf{G}, \mathbf{T})),$$

equipped with the bijection $\vee$ is a root datum, independent of the choice of maximal split torus $\mathbf{T}$.

We will use the following groups associated to a root datum $\mathcal{R}$.

Definition 1.2.6. *Let $\mathcal{R} = (X^*, \Phi, X_*, \Phi^\vee)$ be a root datum. We define the finite Weyl group $W(\Phi)$ of $\mathcal{R}$ to be the group generated by s_a for all $a \in \Phi$, and $W(\mathcal{R}) = X^* \rtimes W(\Phi)$ to be the extended affine Weyl group of $\mathcal{R}$. We let W_{aff} be the affine Weyl group of $\mathcal{R}$.*

Example 1.2.7. *Let $G = GL_2(F)$ and T the split maximal torus of diagonal matrices. Then,*

- *$X^*(T) \cong \mathbb{Z}^2$ with basis e_1, e_2 where*

$$e_i\left(\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right) := a_i$$

- *$X_*(T) \cong \mathbb{Z}^2$ with basis f_1, f_2 where*

$$f_1(x) := \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \; f_2(y) := \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$$

- *$\Phi(G,T) = \{e_1 - e_2, e_2 - e_1\}, \Phi^\vee(G,T) = \{f_1 - f_2, f_2 - f_1\}$.*

- *Let $a = e_1 - e_2$. Then, $a^\vee = f_1 - f_2$. Indeed,*

$$a(a^\vee(x)) = a\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right) = x^2$$

- *$W(\Phi) \cong S_2$.*

- *$W(\mathcal{R}) = \mathbb{Z}^2 \rtimes S_2$.*

Finite Weyl groups and affine Weyl groups are concrete examples of Coxeter groups.

Definition 1.2.8. *A pair (W, S) where W is a group and $S = \{s_1, \dots, s_n\}$ is a finite subset of W is called a Coxeter group if W is generated by S and admits a presentation*

$$W = \langle s_1, s_2, \dots, s_n \in S \mid (s_i s_j)^{m(i,j)} = 1 \rangle,$$

where $m(i,j) \in \mathbb{N} \cup \{\infty\}$, $m(i,i) = 1$ and $m(i,j) \geq 2$. If $m(i,j) = \infty$, then no relation is imposed between s_i and s_j. The numbers $m(i,j)$ are called the Coxeter constants of the group.

The choice of the generators S also gives rise to an important function on W.

Definition 1.2.9. *The length function $l : W \to \mathbb{N}$ of (W, S) assigns to an element $w \in W$ the length $l(w)$ of the minimum expression $w = s_1 \cdots s_{l(w)}$ representing w as a product of elements in S.*

Theorem 1.2.10. *[Spr, Theorem 16.4.2] Split connected reductive groups are uniquely determined by their root datum.*

Quasi-split groups can be characterized by the action of $\Gamma := \mathrm{Gal}(F_{\mathrm{sep}}/F)$ in the Dynkin diagram of the root datum, which is trivial in the split case.

Two algebraic groups $\mathbf{G}, \mathbf{H}$ are called forms of each other if they are isomorphic over F_{sep}. If $\gamma : \mathbf{H} \xrightarrow{\cong} \mathbf{G}$ is an isomorphism, we define an 1-cocycle $\phi_\gamma : \Gamma \to \mathrm{Aut}(\mathbf{G})$ defined by

$$\phi_\gamma(\sigma) := \gamma\sigma\gamma^{-1}\sigma^{-1}.$$

Two isomorphic forms will give cohomologous cocycles, so forms are parametrized up to isomorphism by the Galois cohomology group $H^1(F, \mathrm{Aut}(\mathbf{G}))$. We let $\mathrm{Inn}(\mathbf{G})$ be the subgroup of inner automorphisms of $\mathbf{G}$, which is canonically isomorphic to the adjoint form $\mathbf{G}_{\mathrm{ad}} := \mathbf{G}/Z(\mathbf{G})$ of $\mathbf{G}$.

Definition 1.2.11. *If ϕ_γ takes values in $\mathbf{G}_{\mathrm{ad}}$, $\mathbf{H}$ is called an inner form of $\mathbf{G}$.*

When we also fix an isomorphism of algebraic groups $\gamma : \mathbf{H} \xrightarrow{\cong} \mathbf{G}$ defined over F_{sep} such that $\mathrm{im}(\phi_\gamma) \subseteq \mathbf{G}_{\mathrm{ad}} := \mathbf{G}/Z(\mathbf{G})$, $(\mathbf{H}, \gamma)$ is called an inner twist of $\mathbf{G}$.

Remark 1.2.12. *Two inequivalent inner twists can have isomorphic underlying groups.*

Proposition 1.2.13. *[Spr, Proposition 16.4.9] Every connected reductive group is the inner twist of a quasi-split group.*

Let $Z(G)$ denote the center of G. If $G_{\mathrm{ad}} := G/Z(G)$ is the adjoint form of G, inner twists are parametrized in a canonical way by the Galois cohomology group $H^1(F, G_{\mathrm{ad}})$, if we fix the identity element to correspond to the quasi-split form.

1.2.2 Bernstein theory

Let $P = MU$ be a parabolic subgroup of G with Levi M, and σ a representation of M. We consider the adjoint functors of parabolic induction and restriction $i_M^G : \mathcal{M}(M) \to \mathcal{M}(G), r_G^M :$

$\mathcal{M}(G) \to \mathcal{M}(M)$.

Definition 1.2.14. *A representation σ of M is called supercuspidal if $r_M^N(\sigma) = 0$ for any proper Levi subgroup $N \subseteq M$.*

Lemma 1.2.15. *Every smooth irreducible representation $\pi \in \mathrm{Irr}(G)$ can be embedded in an induced representation $\pi \hookrightarrow i_M^G(\sigma)$ where M is a Levi subgroup of G and σ is a supercuspidal representation of M.*

Proof. Let M be a minimal Levi for G such that $r_G^M(\pi) \neq 0$, and σ be an irreducible quotient of $r_G^M(\pi)$. By transitivity of restriction, $r_G^M(\pi)$ is supercuspidal and therefore σ is supercuspidal. By Frobenius reciprocity, the map $r_G^M(\pi) \twoheadrightarrow \sigma$ provides us a non-trivial map $\pi \to i_M^G(\sigma)$. By irreducibility of π, this map is injective. $\square$

Let $\mathrm{Irr}^c(M)$ be the set of supercuspidal irreducible representations of M. There is an action of the group $X_{\mathrm{nr}}(M) \coloneqq \mathrm{Hom}(M(F)/M(\mathcal{O}), \mathbb{C})$ of unramified characters of M on $\mathrm{Irr}^c(M)$ by twisting given by $\chi \cdot (M, \sigma) = (M, \chi \otimes \sigma)$. Let $\mathfrak{s} = [M, \sigma] \in \mathfrak{B}(G)$ be the orbit of the equivalence class of the cuspidal pair (M, σ) under the action of $X_{\mathrm{nr}}(M)$. We denote by $X_{\mathrm{nr}}(M, \sigma)$ the finite subgroup of $X_{\mathrm{nr}}(M)$ stabilizing σ as an irreducible representation of M. The quotient $T_{\mathfrak{s}} \coloneqq X_{\mathrm{nr}}(M)/X_{\mathrm{nr}}(M, \sigma) \cong \mathrm{Irr}^c(M)$ is a torus, and parametrizes cuspidal pairs with Levi subgroup M up to M-conjugation. Up to G-conjugation, an element $g \in G$ stabilizing a cuspidal pair has to stabilize M, therefore it is in the normalizer $N_G(M)$. As we have already accounted for M-conjugation, G-conjugation becomes the natural action of the Weyl group $W(M) \coloneqq N_G(M)/M$. We denote by $W_{\mathfrak{s}} \leq W(M)$ the finite subgroup stabilizing a cuspidal pair.

Proposition 1.2.16. *The algebraic variety $\Omega_{\mathfrak{s}}(G) = T_{\mathfrak{s}}/W_{\mathfrak{s}}$ parametrizes the equivalence classes of cuspidal pairs in $\mathfrak{s}$.*

Proof. By the discussion of the previous paragraph, equivalence classes in $\mathfrak{s}$ are parametrized by $T_{\mathfrak{s}}$ up to the transitive action of $W_{\mathfrak{s}}$. $\Omega_{\mathfrak{s}}$ admits a natural structure of an algebraic variety since it is the quotient of a torus by a finite group. □

Definition 1.2.17. *Let $\mathfrak{B}(G)$ be the set of all such orbits $\mathfrak{s}$ for all choices of non-conjugate Levi subgroups $M \leq G$. We define the disjoint union*

$$\Omega(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \Omega_{\mathfrak{s}}(G)$$

to be the Bernstein variety of G.

Remark 1.2.18. *Strictly speaking $\Omega(G)$ is not a variety due to the infinite number of connected components, but an infinite union of such. We will often refer to $\Omega_{\mathfrak{s}}$ as a Bernstein variety, but it will be clear from context.*

Proposition 1.2.16 and Definition 1.2.17 imply the following.

Theorem 1.2.19. *The Bernstein variety $\Omega(G)$ parametrizes the set of cuspidal pairs up to G-conjugacy for G. The map $\mathbf{sc} : \mathrm{Irr}(G) \to \Omega(G)$ sending an irreducible representation π to its cuspidal support $\mathbf{sc}(\pi)$ is finite-to-one. In particular, the Bernstein variety parametrizes $\mathrm{Irr}(G)$ in a finite-to-one way.*

We define $\mathrm{Irr}^{\mathfrak{s}}(G) = \mathbf{sc}^{-1}(\Omega_{\mathfrak{s}}(G))$, and $\mathcal{M}_{\mathfrak{s}}(G)$ to be the full subcategory of $\mathcal{M}(G)$ with set of objects $\{V \in \mathcal{M}(G) \mid JH(V) \subseteq \mathrm{Irr}^{\mathfrak{s}}(G)\}$, where $JH(V)$ is the set of Jordan-Holder consituents of V. We can now state the Bernstein decomposition theorem.

Theorem 1.2.20. *[BD84, Proposition 2.10] The partition*

$$\mathrm{Irr}(G) = \bigsqcup_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G)$$

induces a splitting of the abelian category

$$\mathcal{M}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{M}_{\mathfrak{s}}(G).$$

1.2.3 Hecke algebras

The *Hecke algebra of G* is the non-unital algebra $\mathcal{H}(G)$ of locally constant compactly supported distributions on G under convolution. For a smooth representation $\pi \in \mathcal{M}(G)$ with underlying vector space V_π and a fixed vector $v \in V_\pi$ we define a function $f_v : G \to V_\pi$ by $f_v(g) = \pi(g)v$. We define a functor $F : \mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$ sending a smooth representation $\pi \in \mathcal{M}(G)$ with vector space V to the $\mathcal{H}(G)$-module $F(\pi) := V$ with action given by $F(\mathcal{E})v = \langle \mathcal{E}, f_v \rangle$. A module M over a non-unital algebra R is called non-degenerate if $Ann(x) \neq R$ for all $x \in M$.

Proposition 1.2.21. *The functor F defines an equivalence of categories*

$$\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H}(G)),$$

where $\mathcal{M}(\mathcal{H}(G))$ denotes the category of non-degenerate $\mathcal{H}(G)$-modules.

Let $K \leq G$ be a compact open subgroup, $\mathcal{M}_K(G)$ the full subcategory of modules generated by K-fixed vectors, and e_K the normalized constant distribution on K.

Definition 1.2.22. *We define $\mathcal{H}_K(G) := e_K * \mathcal{H} * e_K \subseteq \mathcal{H}(G)$ to be the subalgebra of K-biinvariant distributions.*

We have $\mathcal{H}(G) = \bigcup_K \mathcal{H}_K(G)$.

Proposition 1.2.23. *The functor F restricts to an equivalence of categories*

$$\mathcal{M}_K(G) \cong \mathcal{M}(\mathcal{H}_K(G)).$$

We recall a useful lemma allowing us to represent certain abelian categories as categories of modules over an algebra.

Definition 1.2.24. *A projective generator in an abelian category $\mathcal{M}$ is a projective object Π such that the functor $F_\Pi : \mathcal{M} \to \mathrm{Sets}$ defined by $F_\Pi(X) = \mathrm{Hom}(\Pi, X)$ is faithful and preserves direct sums.*

Lemma 1.2.25. *[Hym68, Theorem 1.3] Let $\mathcal{M}$ be an abelian category with arbitrary direct sums that has a finitely generated projective generator Π. Let $\Lambda = End_{\mathcal{M}}(\Pi)$ be the algebra of endomorphisms of Π in $\mathcal{M}$. Then*

$$\mathcal{M} \cong {}^r\mathcal{M}(\Lambda)$$

the category of right Λ-modules.

Definition 1.2.26. *We call two algebras $\mathcal{H}_1, \mathcal{H}_2$ Morita equivalent if*

$$\mathcal{M}(\mathcal{H}_1) \cong \mathcal{M}(\mathcal{H}_2).$$

We call $\mathcal{H}_1, \mathcal{H}_2$ almost Morita equivalent if

$$\mathcal{M}^{\mathrm{f}}(\mathcal{H}_1) \cong \mathcal{M}^{\mathrm{f}}(\mathcal{H}_2).$$

where $\mathcal{M}^{\mathrm{f}}(\mathcal{H})$ is the category of finite-dimensional modules. We denote Morita equivalence by $\mathcal{H}_1 \sim \mathcal{H}_2$ and almost Morita equivalence by $\mathcal{H}_1 \overset{a}{\sim} \mathcal{H}_2$.

Remark 1.2.27. *A finitely generated projective generator is not unique and different algebras $\mathcal{H}_1, \mathcal{H}_2$ produced by Lemma 1.2.25 do not need to be isomorphic. Instead, since $\mathcal{M}(\mathcal{H}_1) \cong \mathcal{M} \cong \mathcal{M}(\mathcal{H}_2)$, we always have the Morita equivalence*

$$\mathcal{H}_1 \sim \mathcal{H}_2.$$

Let (M, σ) be a representative for $\mathfrak{s} \in \mathfrak{B}(G)$. If A_M is the maximal split torus in $Z(M)$

and M° the maximal compact open of M. Then,

$$\sigma\mid_{M^\circ} = \sigma_1 \oplus \ldots \oplus \sigma_k,$$

with each σ_i irreducible and supercuspidal. We define

$$\Pi_G^{\mathfrak{s}} := \mathrm{ind}_{A_M M^\circ}^{M} \sigma_1,$$

where ind is the functor of compact induction.

Proposition 1.2.28. $\Pi_G^{\mathfrak{s}}$ *is a finitely generated projective generator of* $\mathcal{M}_{\mathfrak{s}}(G)$. *In particular, if* $\mathcal{H}_{\mathfrak{s}} := End_{\mathcal{M}}(\Pi_G^{\mathfrak{s}})$, *we have*

$$\mathcal{M}_{\mathfrak{s}}(G) \cong \mathcal{M}(\mathcal{H}_{\mathfrak{s}}(G)).$$

Proof. The first assertion is [Ber92, Theorem 23], see also [Roc02, Section 1.6]. The second follows by Lemma 1.2.25 and the observation that by the splitting provided by the Bernstein decomposition theorem

$$End_{\mathcal{M}}(\Pi_G^{\mathfrak{s}}) \cong Hom_{\mathcal{M}}(\Pi_G^{\mathfrak{s}}, \Pi_G^{\mathfrak{s}}) \cong Hom_{\mathcal{M}_{\mathfrak{s}}}(\Pi_G^{\mathfrak{s}}, \Pi_G^{\mathfrak{s}}) \cong End_{\mathcal{M}_{\mathfrak{s}}}(\Pi_G^{\mathfrak{s}}).$$

□

We can often get more explicit Hecke algebras via the theory of types [BK98]. Let K be a compact open subgroup of G and ρ a smooth representation of K.

Definition 1.2.29. *The ρ-spherical Hecke algebra $\mathcal{H}(G,\rho)$ of G is the algebra of compactly supported functions*

$$\mathcal{H}(G,\rho) := \{f : G \to End(V_{\tilde{\rho}}) \mid f(kgk') = \tilde{\rho}(k) f(g) \tilde{\rho}(k'),\ \forall k, k' \in K, g \in G\},$$

under convolution with respect to the Haar measure.

Definition 1.2.30. *We define the idempotent*

$$e_\rho(g) = \begin{cases} \frac{dim(V_\rho)}{\mu(K)} tr(\rho(g^{-1})) & \text{if } g \in K \\ 0 & \text{if } g \in G \smallsetminus K. \end{cases}$$

and $\mathcal{H}_\rho := e_\rho \star \mathcal{H} \star e_\rho$. *If* $V \in \mathcal{M}(G)$, *we define* $V_\rho := V * e_\rho \in \mathcal{M}(\mathcal{H}_\rho)$. *For a Bernstein component* $\mathfrak{s} \in \mathfrak{B}(G)$, *the pair* (K, ρ) *is called an* $\mathfrak{s}$*-type if* $\mathbf{sc}(JH(V)) \subseteq \mathfrak{s} \implies V = V_\rho$.

Proposition 1.2.31. *If* (K, ρ) *is an* $\mathfrak{s}$*-type, then*

$$\mathcal{H}_\mathfrak{s} \sim \mathcal{H}(G, \rho) \sim \mathcal{H}_\rho.$$

Proof. By [BK98] there is a canonical isomorphism

$$\mathcal{H}(G, \rho) \otimes End(V_\rho) \xrightarrow{\cong} \mathcal{H}_\rho$$

which by the characterization of Morita equivalence implies

$$\mathcal{H}(G, \rho) \sim \mathcal{H}_\rho.$$

The latter equivalence follows from [BK98, §4]. □

1.2.4 Iwahori-Hecke algebras

The material in this section is based on the excellent exposition [HKP09]. Let $G(F)$ be a split reductive group over a non-archimedean field F with ring of integers $\mathcal{O}$, uniformizer π, and $q = \mathcal{O}/\pi\mathcal{O}$ the cardinality of the residue field. Let T be a maximal split torus and $B = TN$ a Borel. An Iwahori subgroup $I \subseteq G(F)$ is defined to be the preimage of a Borel

$B(k)$ under the natural surjection $G(\mathcal{O}) \twoheadrightarrow G(k)$. $\mathcal{H}_I(G)$ is called the Iwahori-Hecke algebra of G. If $\mathfrak{s}$ is the unramified component of $\Omega(G)$, $\mathcal{H}_{\mathfrak{s}} \cong \mathcal{H}_I(G)$.

In this subsection, we focus more concretely on the Iwahori-Hecke algebra. To ease notation, we set $H := \mathcal{H}_I(G)$. We will show H admits a presentation due to Bernstein, which essentially shows it is isomorphic to an affine Hecke algebra. To do this, we need to define the intertwining operators, which will also be used in Section 5 for the computation of discriminants.

We define $R = \mathbb{C}[X_*(T)] \cong \mathcal{O}(\hat{T})$ to be the group algebra of the cocharacter lattice. It will turn out that R can be embedded in H. First, notice that by sending $\mu \in X_*(T)$ to $\pi^\mu = \mu(\pi) \in T(F)$ we get an isomorphism $X_*(T) \cong T(F)/T(\mathcal{O})$. Using this isomorphism, we view R as a representation of T and thus we can consider the normalized induction $i_B^G(R)$. We define $M = (i_B^G(R))^I$ to be the module of I-fixed vectors, thus M is an $\mathcal{H}_I$-module. By definition, $M \cong C_c(T(\mathcal{O})N\backslash G/I)$. Since $\widetilde{W} \cong T(\mathcal{O})N\backslash G/I$, by setting $v_x = 1_{T(\mathcal{O})NxI}$ we get a $\widetilde{W}$-basis of M as a vector space. Thus, we can define a left R-action on M by $\pi^\mu \cdot v_x = q^{-\langle \rho,\mu\rangle} v_{\pi^\mu \cdot x}$, where ρ is the half-sum of roots of T in $\mathrm{Lie}(N)$.

M is therefore an (R,H)-bimodule. The next proposition is essential.

Proposition 1.2.32. *[HKP09, Lemma 1.6.1] The map $h \to v_1 h$ is an isomorphism of right H-modules from H to M. In particular, $H \cong End_H(M)$.*

Remark 1.2.33. *M is the projective generator of the previous subsection for the unramified Bernstein component.*

Using Proposition 1.2.32, and the left R-action on M, we get an injective morphism $\theta : R \hookrightarrow H$ defined by $rv_1 = v_1\theta(r)$. This allows us to identify R with a subalgebra of H. Even more, $H_W = C_c(I\backslash K/I)$ is the finite Hecke algebra associated to the finite Weyl group W, and it is also a subalgebra of H.

Proposition 1.2.34. *[HKP09, §1.1] The map $f : R \otimes H_W \to H$ defined by $r \otimes h \to \theta(r)h$ is an isomorphism of vector spaces and embeds R, H_W as subalgebras.*

All that is left to have a complete presentation of H is to determine how R and H_W interact. Using certain integrals with respect to the Haar measure, we can define an operator I_{s_a} between two suitable completions of M that satisfies $I_{s_a}(r\phi) = s_a(r)I_{s_a}(\phi)$, see [HKP09, Section 1.10]. After multiplying with $1 - \pi^{-a^\vee}$, we get an operator $J_{s_a} = (1 - \pi^{-a^\vee})I_{s_a} \in \mathrm{End}_H(M)$. By Proposition 1.2.32, J_{s_a} corresponds to an element $j_a \in H$ such that $\pi^\mu j_a = j_a\pi^{s_a(\mu)}$.

The Gindikin-Karpelevich formula [HKP09, Lemma 1.13.1] implies that $J_{s_a}(v_1) = q^{-1}(1-\pi^{a^\vee})v_{s_a} + (1-q^{-1})\pi^{a^\vee}v_1$, thus by proposition 1.2.32 we get $j_a = q^{-1}(1-\pi^{a^\vee})T_{s_a} + (1-q^{-1})\pi^{a^\vee}$ since the actions of the two elements on v_1 agree.

Notice that I_{s_a} does not correspond to an element of H essentially because it has denominators in R. We remedy this problem by considering L to be the function field of R and then defining $H_L = H \otimes_R L$. In H_L, we can perform computations with denominators, and we have elements i_a corresponding to the intertwining operators where the previous relation becomes

$$i_a = q^{-1}T_{s_a} + \frac{(1-q^{-1})\pi^{a^\vee}}{1-\pi^{a^\vee}}$$

To ease computations, we define the elements

$$c_a = \frac{e_a}{d_a} = \frac{1 - q^{-1}\pi^{a^\vee}}{1-\pi^{a^\vee}} \in L.$$

Now we will derive the quadratic relation satisfied by i_a from the quadratic relation $(T_{s_a} - q)(T_{s_a} + 1) = 0$. We have:

$$\begin{aligned} T_{s_a} - q &= q\,(i_a - c_a) \\ T_{s_a} + 1 &= q\,(i_a + s_a(c_a)) \end{aligned}$$

which gives us

$$i_a^2 = c_a s_a(c_a) \tag{1.2}$$

By induction on the length of w, we get the following Lemma that will be useful in Section 5.

Lemma 1.2.35. *Let $w \in W$ and $R_w = \{a \in \Delta \mid a > 0, w(a) < 0\}$. Then,*

$$I_w I_{w^{-1}} = \prod_{a \in R_w} \frac{e_a e_{-a}}{d_a d_{-a}}.$$

We can also derive the intertwining relation satisfied by the T_{s_a} from the intertwining relation $i_a r = s_a(r) i_a$. Combining with the quadratic relation, we get the Bernstein presentation for the Iwahori-Hecke algebra [HKP09, §1]

Proposition 1.2.36. *The Iwahori-Hecke algebra of G is generated over the group algebra of the cocharacter lattice by elements $T_w = T_{s_1} \cdots T_{s_n}$ where $w = s_1 \cdots s_n$ is a reduced expression for w, and for a simple root a the element T_{s_a} satisfies*

$$(T_{s_a} + 1)(T_{s_a} - q) = 0 \tag{1.3}$$

$$T_{s_a} \pi^{\mu} = \pi^{s_a(\mu)} T_{s_a} + \frac{(q-1)(\pi^{\mu} - \pi^{s_a(\mu)})}{1 - \pi^{-a^\vee}}. \tag{1.4}$$

We can also use the intertwining elements, to determine the center of H. From the intertwining relations, it follows that W-invariant elements of R are in the center, ie. $R^W \subseteq Z(H)$. The other direction is also true by virtue of the Satake isomorphism.

Indeed, given the relation 1.2, it makes sense to define the normalized intertwining operators $k_a = i_a c_a^{-1}$, since then

$$k_a^2 = i_a c_a^{-1} i_a c_a^{-1} = \frac{i_a^2}{s_a(c_a) c_a} = 1 \tag{1.5}$$

Lemma 1.2.37. *There exist elements $k_w \in H_L^I$ indexed by $w \in W$ which form a basis of H_L^I as L-vector space such that $k_{s_\alpha} = k_\alpha$ for all simple coroots α, $k_{ww'} = k_w k_{w'}$ for all $w, w' \in W$ and $k_w l = w(l) k_w$. In other words, we have an isomorphism between H_L^I with the twisted group algebra $L\langle W \rangle$.*

Proof. We will prove that there exists a unique morphism of groups $W \to (H_L^I)^\times$ given by $w \mapsto k_w$ such that $k_{s_\alpha} = k_\alpha$ for all simple coroots. Since the Weyl group W is the Coxeter group defined by reflections s_α and braid relations $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$, integers $m(\alpha,\beta)$ being the Cartan constants associated with the root system, we only have to prove that that the involution k_α satisfies the same braid relations $(k_\alpha k_\beta)^{m(\alpha,\beta)} = 1$. If $w = s_{\alpha_1} \dots s_{\alpha_n}$ is an expression of $w \in W$ as the product of simple reflections s_{α_i}, then $k_w = k_{\alpha_1} \dots k_{\alpha_n}$ is then an invertible element of H_L^I depending only on w. In particular, we will then have $k_{ww'} = k_w k_{w'}$ for all $w, w' \in W$. We know that the elements k_w so defined satisfy the commutation relation $k_w l = w(l) k_w$. We will also prove that the elements k_w for a basis of L-vector space H_L^I.

For every $w \in W$, let H_L^w denote the L-vector space of elements $h \in H_L^I$ such that $hl = w(l)h$ for all $l \in L$. If $w = s_{\alpha_1} \dots s_{\alpha_n}$ is an expression of an element $w \in W$ as a product of simple reflections s_{α_i}, then the product $k_{\alpha_1} \dots k_{\alpha_n}$ is an invertible element of H_L lying in H_L^w. It follows that $\dim(H_L^w) \geq 1$.

Next we prove that the subspaces H_L^w of H_L are linearly independent. We will prove that if $\sum_{w \in W} h_w = 0$ with $h_w \in H_L^w$ then $h_w = 0$ for every $w \in W$. For this we need some basic facts in Galois theory. We recall that L is a Galois extension of $K = L^W$ of Galois group W. We have distinct morphisms of algebras $L \otimes_k L \to L$ given by $x \otimes y \to x\sigma(y)$ for each $w \in W$. This gives rise to an isomorphism of L-algebras $L \otimes_k L \to L^W$ which is in particular an isomorphism of L-vector spaces. Concretely, for every basis $l_1, \dots, l_m$ of L as a vector space over K, the vectors $(w(l_i), w \in W) \in L^W$ form a basis of L^W. Now, for every $i \in \{1, \dots, m\}$, we have

$$0 = \sum_{w \in W} h_w l_i = \sum_{w \in W} w(l_i) h_w.$$

Since the vectors $(w(l_i), w \in W) \in L^W$ form a basis of L^W, it follows that $h_w = 0$ for every $w \in W$.

Since $\dim_L(H_L^w) = |W|$, we derive $\dim(H_L^W) = 1$ for all $w \in W$, and

$$H_L^I = \bigoplus_{w \in W} H_L^w.$$

Now, if α, β are distinct simple coroots and $(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1$ then $(k_\alpha k_\beta)^{m(\alpha,\beta)} \in H_L^1$ therefore $(k_\alpha k_\beta)^{m(\alpha,\beta)} \in L$. We define $v_0^K = 1_K \in C_c^\infty(K \backslash G/K)$. Combining with the spherical Gindikin-Karpelevich formula $v_0^K k_\alpha = v_0^K$ we have $v_0^K (k_\alpha k_\beta)^{m(\alpha,\beta)} = v_0^K$. It follows that $(k_\alpha k_\beta)^{m(\alpha,\beta)} = 1$. □

Theorem 1.2.38 (Satake isomorphism)**.** *The center $Z(H)$ of H is $R^W \cong \mathcal{O}(\hat{T} /\!\!/ W)$.*

Proof. Since $L\langle W\rangle$ is a matrix algebra over L^W, its center consists of the scalar matrices, ie. L^W. Thus, the center of H is

$$H \cap L^W = R^W.$$

□

1.2.5 Affine Hecke algebras

The results of the previous subsection admit generalizations in the sense of providing specific presentations for algebras that are (almost) Morita equivalent to Hecke algebras of Bernstein components. Affine Hecke algebras have since found applications in various areas of mathematics, such as knot theory, combinatorics, representations of finite groups, etc. Due to the fact they admit a Bernstein presentation, Theorem 1.1.1 can be applied. We recommend the exposition [Sol21].

Proposition 1.2.39. *Let (W, S) be a Coxeter group, equipped with a function $q : S \to \mathbb{C}$ such that $q(s) = q(s')$ if s, s' are conjugate in W. There is a unique algebra structure $\mathcal{H}(W, q)$ on* *the vector space over $\mathbb{C}$ generated by elements $T_w, w \in W$ such that*

- $T_e = 1$,
- $(T_s - q(s))(T_s + 1) = 0, s \in S$,
- $T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots$ *where both sides have* $m(s, s')$ *elements,*
- $T_{w_1 w_2} = T_{w_1} T_{w_2}$ *if* $l(w_1 w_2) = l(w_1) + l(w_2)$.

We call $\mathcal{H}(W, q)$ the Iwahori-Hecke algebra of (W, q). If W is finite, we call $\mathcal{H}(W, q)$ its finite Hecke algebra. If (W, S) is an affine Weyl group, we say $\mathcal{H}(W, q)$ is of affine type.

The connection with the definitions of the previous subsection is the following theorem [IM65].

Proposition 1.2.40 (Iwahori-Matsumoto presentation)**.** *If G is a split, simply connected, semisimple group over $\mathbb{Q}_p$, T a split maximal torus, and (W, S) the affine Weyl group of the dual root datum $\mathcal{R}(G, T)^\vee$. Let $q(s) = p, \ \forall s \in S$. Then,*

$$\mathcal{H}_I(G) \cong \mathcal{H}(W, q).$$

Affine Hecke algebras are a generalization of Iwahori-Hecke algebras of affine type.

Proposition 1.2.41. *Let $\mathcal{R} = (X^*, \Phi, X_*, \Phi^\vee)$ be an irreducible root datum with finite Weyl group W, $q \in \mathbb{R}_{\geq 1}$, and $\lambda, \lambda^* : \Phi \to \mathbb{C}$ be W-invariant functions such that*

$$a^\vee \notin 2X_* \implies \lambda(a) = \lambda^*(a).$$

Let $\mathbb{C}[X^*]$ be the group algebra of the character lattice, with the standard basis $\{\theta_x, x \in X^*\}$ and $\mathcal{H}(W, q)$ the finite Hecke algebra of W.

There is a unique algebra structure on the vector space $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) := \mathbb{C}[X^*] \otimes \mathcal{H}(W, q)$ such that the following are true.

- *$\mathbb{C}[X^*], \mathcal{H}(W,q)$ are embedded as subalgebras.*

- *For $a \in \Delta, x \in X$*

$$\theta_x T_{s_a} - T_{s_a}\theta_{s_a(x)} = \left((q^{\lambda(a)} - 1) + \theta_{-a}\left(q^{(\lambda(a)+\lambda^*(a))/2} - q^{(\lambda(a)-\lambda^*(a))/2}\right)\right)\frac{\theta_x - \theta_{s_a(x)}}{\theta_0 - \theta_{-2a}}.$$

We call $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,q)$ the affine Hecke algebra of $\mathcal{R}$.

If $\lambda(a) = \lambda(b) = \lambda^*(a) = \lambda^*(b)$ for all a,b, we say $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,q)$ has equal parameters.

Remark 1.2.42. *Notice that if $a^\vee \notin 2X_*$ or if $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,q)$ has equal parameters, the second relation simplifies to*

$$\theta_x T_{s_a} - T_{s_a}\theta_{s_a(x)} = (q^{\lambda(a)} - 1)\frac{\theta_x - \theta_{s_a(x)}}{\theta_0 - \theta_{-a}}.$$

Remark 1.2.43. *If $\mathcal{R}$ is not irreducible, let d be the number of connected components. As noticed in [AMS21] , the proposition remains true if we substitute $\vec{\mathbf{z}} = (\vec{\mathbf{z}}_1, \ldots, \vec{\mathbf{z}}_d)$ in the place of $\mathbf{q}$ with the obvious changes in the relations. We also call this an affine Hecke algebra $\mathcal{H}(\mathcal{R},\lambda,\lambda^*,\vec{\mathbf{z}})$.*

Not every affine Hecke algebra is an Iwahori-Hecke algebra, but we do have a generalization of the Iwahori-Matsumoto presentation. Let $\Omega := \{w \in W(\mathcal{R}) \mid l(w) = 0\}$. Then,

$$W(\mathcal{R}) = W_{\text{aff}} \rtimes \Omega.$$

Proposition 1.2.44. *There is a unique algebra isomorphism*

$$\mathcal{H}(\mathcal{R},\lambda,\lambda^*,q) \xrightarrow{\cong} \mathcal{H}(W,q) \rtimes \Omega$$

such that

- *It restricts to the identity in $\mathcal{H}(W,q)$.*

- *For all* $x \in \mathbb{Z}\Phi$ *with* $\langle x, a^\vee \rangle \geq 0, \ \forall \ a \in \Delta$, *it sends* θ_x *to* $q(x)^{-1/2} T_x$.

For many Bernstein components $\mathfrak{s} \in \mathfrak{B}(G)$, $\mathcal{H}_\mathfrak{s}$ is Morita equivalent to an affine Hecke algebra. In [Sol22], it was proven that one slight generalization is still needed.

Definition 1.2.45. *Consider the following data:*

1. *A root datum* $\mathcal{R} = (X^*, \Phi, X_*, \Phi^\vee)$ *with simple roots* Δ,
2. *A finite group* $\mathfrak{R}$ *acting on* $W(\mathcal{R})$,
3. *A* 2*-cocycle* $\natural : (W/W(R))^2 \to \mathbb{C}$, *where* $W = W(R) \rtimes \mathfrak{R}$.
4. W*-invariant functions* $\lambda, \lambda^* : \Phi \to \mathbb{C}$, *such that* $a^\vee \notin 2X_* \implies \lambda(a) = \lambda^*(a)$.
5. *An array of invertible elements* $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_d)$.

We define the twisted affine Hecke algebra to be $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}}) \rtimes \mathbb{C}[\mathfrak{R}, \natural]$.

For a choice of parameters $\vec{\mathbf{z}} = (z_1, \ldots, z_d)$, we can specialize a twisted Hecke algebra.

We recall a more geometric construction that ties nicely with our perspective. Let $T :=$ $\mathrm{Hom}(X, \mathbb{C}^\times)$ a complex algebraic torus. Since $\mathcal{O}(T) \cong \mathbb{C}[X]$, the group W acts naturally on T.

Proposition 1.2.46. *There is a unique algebra structure on the vector space*

$$\mathcal{H}(T, W, \lambda, \lambda^*, \natural, \vec{\mathbf{z}}) := \mathcal{O}(T) \otimes \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}] \otimes \mathbb{C}[W(\mathcal{R})] \otimes \mathbb{C}[\mathfrak{R}, \natural]$$

such that

- *Under the isomorphism* $\mathcal{O}(T) \cong \mathbb{C}[X^*]$, *the span of* $\mathcal{O}(T), \mathbb{C}[\vec{\mathbf{z}}, \vec{\mathbf{z}}^{-1}]$ *and* $\mathbb{C}[W(\mathcal{R})]$ *is the affine Hecke algebra* $\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, \vec{\mathbf{z}})$.
- $\mathbb{C}[\mathfrak{R}, \natural]$ *embeds as a subalgebra.*

- *For $\gamma \in \mathfrak{R}$, $w \in W(R)$ and $x \in \mathcal{O}(T)$:*

$$T_\gamma T_w \theta_x T_\gamma^{-1} = T_{\gamma x \gamma^{-1}} \theta_{\gamma(x)}.$$

If $\mathfrak{R} = 1$, then $\mathcal{H}(T, W, \lambda, \lambda^*, \natural, \vec{\mathbf{z}})$ is the affine Hecke algebra of $\mathcal{R}$.

Proof. Similar to [AMS21, Proposition 2.2]. □

Our motivation for introducing twisted affine Hecke algebras is the next theorem [Sol22].

Theorem 1.2.47. *Let G be a reductive group and $\mathfrak{s} \in \mathfrak{B}(G)$. Then, there exist parameters λ, λ^* and a cocycle $\natural$ such that $\mathcal{H}_\mathfrak{s}$ is almost Morita equivalent to a specialization $H_\mathfrak{s}$ of the twisted affine Hecke algebra $\mathcal{H}(T_\mathfrak{s}, W_\mathfrak{s}, \lambda, \lambda^*, \natural, \vec{\mathbf{z}})$.*

The classical Satake isomorphism admits the following generalization.

Lemma 1.2.48. *[AMS21, Lemma 2.3] $\mathcal{O}(T \times \mathbb{C}^d)^W$ is a central subalgebra of $\mathcal{H}(T, W, \lambda, \lambda^*, \natural, \vec{\mathbf{z}})$. It equals $Z(\mathcal{H}(T, W, \lambda, \lambda^*, \natural, \vec{\mathbf{z}}))$ if W acts faithfully on T. For a specialization H, we have*

$$Z(H) \cong \mathcal{O}(T)^W.$$

Remark 1.2.49. *Since almost Morita equivalence preserves the center, combining Theorem 1.2.47 and Lemma 1.2.48, we retrieve the fact that*

$$\mathcal{O}(\Omega_\mathfrak{s}(G)) \cong Z(H_\mathfrak{s}) \cong Z(\mathcal{H}_\mathfrak{s}(G)).$$

1.3 The Lafforgue variety

The main goal of this Section is to give a proof of Theorem 1.1.1. Let R be a possibly non-commutative A-algebra over a finitely generated commutative central k-subalgebra A such that R is a finite A-module. Lafforgue's original assertion concerns the case of the

Hecke algebra $\mathcal{H}_K$ of a reductive p-adic group G and a compact open subgroup $K \le G$. In particular, Lafforgue's assertion is Theorem 1.1.1 for $R = \mathcal{H}_K$, $A = Z_K$ and $k = \mathbb{C}$.

If M is a $\bar{k}$-finite dimensional simple R-module, then A must act on M through a character $a : A \to \bar{k}$. The Lafforgue variety can thus be thought of as a scheme over $\mathrm{Spec}(A)$. We construct a non-commutative Hilbert scheme for the finite A-algebra R.

Assume now $\mathrm{char}(k) = 0$. Then, we construct a trace map from the Hilbert scheme to a generalized Grothendieck vector bundle. The Lafforgue variety will be defined to be the image of that map.

We use the same strategy in positive characteristic, albeit with a twist. Due to the elementary fact that a simple module is not determined by its traces in positive characteristic, we construct a determinant map based on Roby's concept of a polynomial law [Rob63].

1.3.1 Non-commutative Hilbert scheme

Let A be a commutative ring which is contained in the center of a possibly non-commutative ring R.

Definition 1.3.1. *We call the non-commutative Hilbert functor* $\mathrm{Hilb}_{R/A}$ *the functor that associates to every commutative A-algebra B the set of isomorphism classes of $R \otimes_A B$-modules M, which are flat as B-modules, equipped with a surjective $R \otimes_A B$-linear map $R \otimes_A B \twoheadrightarrow M$.*

Proposition 1.3.2. *The functor* $\mathrm{Hilb}_{R/A}$ *is representable by a proper scheme over* $\mathrm{Spec}(A)$.

Proof. We consider R just as a finite A-module. The Quot functor $\mathcal{Q}_{R/A}$ associating to every commutative A-algebra B the set of isomorphism classes of flat B-modules M equipped with a surjective $R \otimes_A B$-linear map $m : R \otimes_A B \twoheadrightarrow M$ is representable by a projective scheme over $\mathrm{Spec}(A)$, by [Gro61, Théorème 3.1] for $S = Spec(A), X = \mathrm{Spec}(A)$, and we set $T = \mathrm{Spec}(B)$. Since the functor $\mathrm{Hilb}_{R/A}$ is a closed subfunctor of $\mathcal{Q}_{R/A}$, it is also representable by projective scheme over $\mathrm{Spec}(A)$. □

There is a decomposition of $\mathrm{Hilb}_{R/A}$ into open and closed subschemes

$$\mathrm{Hilb}_{R/A} = \bigsqcup_{d\in\mathbb{N}} \mathrm{Hilb}^d_{R/A} \tag{1.6}$$

where $\mathrm{Hilb}^d_{R/A}$ classifies $R \otimes_A B$-linear maps $m : R \otimes_A B \twoheadrightarrow M$ with M being a locally free B-module of rank d.

Since we are mainly interested in irreducible modules, it will also be useful to consider the following functor.

Definition 1.3.3. *The nested non-commutative Hilbert functor* $n\mathrm{Hilb}_{R/A}$ *which associates to every commutative* A*-algebra* B *the set of isomorphism classes of pairs of* $R\otimes_A B$*-modules* M, N*, which are flat as* B*-modules, equipped with surjective* $R \otimes_A B$*-linear maps* $R \otimes_A B \twoheadrightarrow M \twoheadrightarrow N$*, where we also require that the latter map has a non-zero kernel.*

Proposition 1.3.4. *The functor* $n\mathrm{Hilb}_{R/A}$ *is representable by a proper scheme over* $\mathrm{Spec}(A)$.

The proof is the same as $n\mathrm{Hilb}_{R/A}$ is a closed subscheme of a relative flag variety.

Proposition 1.3.5. *The forgetful map*

$$F_N : n\mathrm{Hilb}_{R/A} \to \mathrm{Hilb}_{R/A}$$

defined by $F_N(M, N) := N$ *is a proper morphism. In particular, the complement* $i\mathrm{Hilb}_{R/A}$ *of the image of* F_N *is open.*

Proof. By the fact that $n\mathrm{Hilb}_{R/A}$ is proper over $\mathrm{Spec}(A)$ and $\mathrm{Hilb}_{R/A}$ is separated over $\mathrm{Spec}(A)$, the first part follows from [Sta22, Lemma 01W6]. The second part readily follows. □

A geometric point $x \in i\mathrm{Hilb}_{R/A}(\bar{k})$ over a point $a : A \to \bar{k}$ consists of a quotient M_x of the algebra $R_a = R \otimes_A \bar{k}$ by a maximal left ideal, or in other words, M_x is a simple R_a-module equipped with a generator.

We consider the group scheme $\mathcal{G}_{R/A}$ over $\mathrm{Spec}(A)$ which associates to every commutative A-algebra the group $(R \otimes_A B)^\times$ of invertible elements of the possibly non-commutative algebra $R \otimes_A B$. This group scheme is smooth over $\mathrm{Spec}(A)$ if R is a finite locally free A-module.

The group scheme $\mathcal{G}_{R/A}$ acts on $\mathrm{Hilb}_{R/A}$ relative to $\mathrm{Spec}(A)$. For a B-point of $(M, m) \in \mathrm{Hilb}_{R/A}(B)$ we will denote the action of $R \otimes_A B$ on M by $(r, m) \mapsto e(r)m$. If $g \in (R \otimes_A B)^\times$ we define the action of g on (M, m) to be $g(M, m) = (M', m')$ where $M' = M$ as a B-module equipped with the structure of $R \otimes_A B$-module given by $e'(r)m = e(g^{-1}rg)m$, and $m' = mg$. Similarly, the group scheme $\mathcal{G}_{R/A}$ also acts on the nested Hilbert scheme $n\mathrm{Hilb}_{R/A}$.

Proposition 1.3.6. *The morphism $F_N : n\mathrm{Hilb}_{R/A} \to \mathrm{Hilb}_{R/A}$ is $\mathcal{G}_{R/A}$-equivariant, and the complement of its image $i\mathrm{Hilb}_{R/A}$ of* $\mathrm{Hilb}_{R/A}$, *is open and stable under the action of $\mathcal{G}_{R/A}$.*

Proof. The first part follows from the definition of the action. Therefore, the image is a $\mathcal{G}_{R/A}$-equivariant closed subscheme of $\mathrm{Hilb}_{R/A}$. The second part follows from this observation. □

1.3.2 Trace map

Following Grothendieck, we define a generalized vector bundle $V_{R/A}$ over a commutative ring attached to an A-module R. As a functor, $V_{R/A}$ attaches to each A-algebra B the abelian group $\mathrm{Hom}_A(R, B)$ of all A-linear maps $R \to B$. This functor is represented by the symmetric algebra $\mathrm{Sym}_A(R)$: it is the $\mathbb{N}$-graded A-algebra with $\mathrm{Sym}^0_A(R) = A$, $\mathrm{Sym}^1_A(R) = R$, and for every $d \in \mathbb{N}$, the d-th symmetric power $\mathrm{Sym}^d_A(R)$ is the largest quotient of the dth fold tensor power $R^{\otimes d}$ of R over A on which the symmetric group S_d acts trivially. We claim that the morphism of functors on A-algebras:

$$\mathrm{Hom}_{A-Alg}(\mathrm{Sym}_A(R), B) \to \mathrm{Hom}_A(R, B),$$

defined as the restriction an A-algebra homomorphism $x : \mathrm{Sym}_A(R) \to B$ to the degree 1 component $\mathrm{Sym}_A^1(R) = R$, is an isomorphism of functors. Indeed, every A-linear map $y : R \to B$, induces an A-linear map $R^{\otimes d} \to B^{\otimes d} \to B$ which factors through an A-linear map $y^d : \mathrm{Sym}_A^d(R) \to B$. It's not hard to check that the A-linear map $x : \bigoplus_{d\in\mathbb{N}} \mathrm{Sym}_A^d(R) \to B$ given by $x = \bigoplus_{d\in\mathbb{N}} y^d$ is a homomorphism of A-algebras. It is also clear that the map $y \mapsto x$ thus defined gives rise to an inverse of the functor $x \mapsto y$. We conclude that the functor $V_{R/A}$ is representable by the affine scheme $\mathrm{Spec}(\mathrm{Sym}_A(R))$ which is a generalized vector bundle in the sense of Grothendieck.

We assume that R is a finite A-module which is equipped with a structure of a possibly non-commutative algebra containing A in its center. We can construct the trace map

$$\mathrm{tr}_{R/A} : \mathrm{Hilb}_{R/A} \to V_{R/A} \tag{1.7}$$

as follows. For every point $(M, m) \in \mathrm{Hilb}_{R/A}(B)$, where M is an $R \otimes_A B$-module that is locally free and finite as a B-module. Every $r \in R$ defines a B-linear operator of M given by the structure of an $R \otimes_A B$-module. Since M is a finitely generated locally free B-module, the trace $\mathrm{tr}_B(r) \in B$ is well defined. This gives rise to an A-linear map $\mathrm{tr}_M : R \to B$ and thus to a B-point of $V_{R/A}$.

Since $\mathrm{Hilb}_{R/A}$ is proper over $\mathrm{Spec}(A)$, whereas $V_{R/A}$ is affine by construction, there exists a closed subscheme $\mathrm{Laf}_{R/A}$ of $V_{R/A}$, finite over $\mathrm{Spec}(A)$ such that tr factors through a proper surjective map $\mathrm{tr}_L : \mathrm{Hilb}_{R/A} \to \mathrm{Laf}_{R/A}$. Let $\mathrm{iLaf}_{R/A}$ denote the open subscheme of $\mathrm{Laf}_{R/A}$ which is the complement of the closed subset that is defined as the image of the proper map $\mathrm{tr}_L : n\mathrm{Hilb}_{R/A} \to \mathrm{Laf}_{R/A}$.

Proposition 1.3.7. *Assume that A is a k-algebra where k is a field of characteristic zero. Then the preimage $\mathrm{tr}_L^{-1}(\mathrm{iLaf}_{R/A})$ is $i\mathrm{Hilb}_{R/A}$. Moreover for every geometric point $l \in \mathrm{iLaf}_{R/A}(\bar{k})$ over $a : A \to \bar{k}$, the group $\mathcal{G}_{R_a}$ acts transitively on the fiber $\mathrm{tr}_L^{-1}(l)$.*

This assertion is nothing but a reformulation of well known facts about modules over a finite-dimensional algebra, improperly referred to as Brauer-Nesbitt's theorem. As we want to extend the construction of Lafforgue's variety to the case of positive characteristic, we will give a sketch of the proof of Proposition 1.3.8 which was given in full details in Lang's book [Lan02, chapter XVII Cor 3.8] to which we refer for more information.

Proposition 1.3.8 (Bourbaki)**.** *Assume that k is a field of characteristic zero and R is a finite-dimensional k-algebra possibly non-commutative. Let M and N be k-finite dimensional R-modules such that for all $x \in R$, we have $\mathrm{tr}_x(M) = \mathrm{tr}_x(N)$, then M and N have the same semi-simplification. In particular if $\mathrm{tr}_L(x_1) = \mathrm{tr}_L(x_2)$, and if M_1 is a simple R_a-module then M_2 is also simple and $M_2 \cong M_1$.*

Proof. The assertion is obvious in one direction. If the quotients M_1 and M_2 of R_a have the same semi-simplification as R_a-modules then the induced linear forms $\mathrm{tr}_{M_1}, \mathrm{tr}_{M_2} : R_a \to \bar{k}$ are equal because traces only depend of semi-simplification. Conversely, Jacobson's density theorem [Jac45] implies the existence of projectors: if $V_0, V_1, \ldots, V_n$ are non-isomorphic simple R_a-modules, there exists an element $e_0 \in R$ which acts as the identity on V_0 and 0 on $V_1, \ldots, V_n$. Note that Jacobson's density theorem is valid in any characteristic. Now let $V_0, \ldots, V_n$ be simple R_a-modules occurring as a simple subquotient of M_1 or M_2 and write decompositions of the semi-simplifications of M_1 and M_2 as $M_1^{ss} = V_0^{m_1} \oplus U_1$ and $M_2^{ss} = V_0^{m_2} \oplus U_2$ where U_1 and U_2 are semi-simple modules with no occurrences of V_0. To prove that M_1 and M_2 have the same semi-simplification, it is enough to prove $m_1 = m_2$. This derives from the equalities

$$m_1 \dim(V_0) = \mathrm{tr}_{M_1}(e_0) = \mathrm{tr}_{M_2}(e_0) = m_2 \dim(V_0)$$

as elements of $\bar{k}$. The characteristic zero assumption is only used to guarantee that in $\bar{k}$ we have $\dim(V_0) \neq 0$. $\square$

The last statement of Proposition 1.3.8 implies that $\mathrm{tr}_L^{-1}(\mathrm{iLaf}_{R/A}) = i\mathrm{Hilb}_{R/A}$. It also implies that if $l \in \mathrm{iLaf}_{R/A}(\bar{k})$ over $a : A \to \bar{k}$, and if $x_1, x_2 \in \mathrm{tr}_L^{-1}(l)$ are represented by quotients M_1 and M_2 of R_a then M_1 and M_2 are isomorphic simple R_a-modules. It follows that there exists $g \in M_a^\times$ such that $gx_1 = x_2$. In other words, the fiber of $\mathcal{G}_{R/A}$ over a acts transitively on the fiber of tr over l.

We recall and prove Theorem 1.1.1.

Theorem 1.1.1. $\mathrm{Irr}(R)$ *forms the set of* $\bar{k}$*-points of a dense Zariski open subscheme* $\mathrm{iLaf}_{R/A}$ *of* $\mathrm{Laf}_{R/A}$. *The projection* $p : \mathrm{Laf}_{R/A} \to \mathrm{Spec}(A)$ *is finite.*

Proof. By Proposition 1.3.8, the trace map $\mathrm{tr} : \mathrm{Hilb}_{R/A} \to \mathrm{Laf}_{R/A}$ forgets the choice of a generator for the module M and parametrizes modules up to isomorphism. Since $\mathrm{iLaf}_{R/A}$ is the complement of the image $\mathrm{tr} \circ F_N$, modules in $\mathrm{iLaf}_{R/A}$ are the ones not admitting a proper quotient $M \twoheadrightarrow N$, therefore they are simple.

Since $\mathrm{Hilb}_{R/A}$ is proper over $\mathrm{Spec}A$ and $V_{R/A}$ is separated and locally of finite type, p is proper by [Sta22, Tag 0AH6]. Since $\mathrm{Laf}_{R/A}$ is a closed subscheme of the affine scheme $V_{R/A}$, p is affine. Therefore, p is finite. □

Without any hypothesis on the characteristic, we have to replace the trace by the determinant. Let us formalize the construction of the determinant map as an analogue of the trace map previously defined.

1.3.3 Determinant map

Let A be a commutative ring. An A-module R gives rise to a functor $\underline{R} : B \mapsto R \otimes_A B$ from the category of A-algebras to the category of sets. We recall the definition of a polynomial law in [Rob63]

Definition 1.3.9. *A polynomial law on* R *is a morphism of functors* $f : \underline{R} \to \underline{A}$ *where* $\underline{A}$ *is the functor* $B \mapsto A \otimes_A B = B$.

We denote by $\mathrm{Pol}_A(R)$ *the set of all polynomial laws on the* A*-module* R*.*

Thus, a polynomial law f on R consists of a family of set theoretical maps $f_B : R \otimes_A B \to B$ depending on B in a functorial way.

If $r_1, \dots, r_n \in R \otimes_A B$ form a finite sequence $\underline{r}$ of elements of $R \otimes_A B$, then f gives rise to a polynomial $f_{\underline{r}} \in B[X_1, \dots, X_n]$, where $X_1, \dots, X_n$ are free variables, such that for every $x_1, \dots, x_n \in B$, we have $f_{\underline{r}}(x_1, \dots, x_n) = f(x_1 r_1 + \dots + x_n r_n)$. Indeed, if we take

$$X_{\underline{r}} = r_1 \otimes X_1 + \dots + r_n \otimes X_n \in R \otimes_A B[X_1, \dots, X_r],$$

then we set

$$f_{\underline{r}} := f_B(X_{\underline{r}}) \in B[X_1, \dots, X_r],$$

see [Rob63, Thm 1.1]. The main point in Roby's concept of polynomial law is that the polynomial $f_{\underline{r}}$ is a part of the data of f.

Definition 1.3.10. *We say that the polynomial law* $f : \underline{R} \to \underline{A}$ *is homogeneous of degree* $d \in \mathbb{N}$ *if for every* A*-algebra* B*,* $x \in B$ *and* $r \in R \otimes_A B$ *we have* $f_B(xr) = x^d f_B(r)$*.*

We denote by $\mathrm{Pol}_A^d(R)$ *the set of all homogeneous polynomial laws of degree* d *on the* A*-module* R*.*

It's not hard to check that the polynomial law f is homogeneous of degree d if and only if for every finite sequence $\underline{r} = (r_1, \dots, r_n)$ of elements of $R \otimes_A B$ for any A-algebra B, $f_{\underline{r}}$ is a homogeneous polynomial of degree d with coefficients in B, [Rob63, Prop. I.1, p. 226].

Example 1.3.11. *A homogeneous polynomial law of degree* 1 *on* R *consists of a family of linear forms* $f_B : M \otimes_A B \to B$ *depending functorially on* B *which is equivalent to the initial linear form* $f_A : M \to A$*.*

Now, we will generalize Grothendieck's construction of generalized vector bundle associated to an A-module, by replacing linear forms on M by homogeneous polynomial laws of degree d. Let A be a commutative ring and R a A-module.

Definition 1.3.12. *We define the functor $S^dV_{R/A}$ which attaches to every A-algebra B the set $\mathrm{Pol}^d_B(R\otimes_A B)$ of polynomial laws on the B-module $R\otimes B$ which are homogeneous of degree d.*

Example 1.3.13. *For $d=1$, $\mathrm{Pol}^1_B(R\otimes_A B)=\mathrm{Hom}_B(R,B)$ and we have an isomorphism of functors $S^1V_{R/A}=V_{R/A}$ which is represented by the affine scheme $\mathrm{Spec}(\mathrm{Sym}_A(R))$.*

The above example generalizes.

Definition 1.3.14. *[Rob63, Ch. III, p. 249] We define Γ^d_aR to be the d-th divided power of the A-module M.*

Proposition 1.3.15. *[Rob63, Thm II.3 p. 262, IV.1 p. 266] For every $d\in\mathbb{N}$, there is a canonical isomorphism of functors $\mathrm{Pol}^d_B(R\otimes_A B)=\mathrm{Hom}_B(\Gamma^d_AR,B)$.*

We immediately obtain the following proposition.

Proposition 1.3.16. *The functor $B\mapsto S^dV_{R/A}(B)=\mathrm{Pol}^d_B(R\otimes_A B)$ is representable by the affine scheme $\mathrm{Spec}(\mathrm{Sym}_A(\Gamma^d_aR))$.*

Let R be a possibly non-commutative algebra containing a commutative ring A in its center such that R is finite locally free A-module.

Every point $x\in\mathrm{Hilb}^d_{R/A}(B)$ is represented by an $(R\otimes_A B)$-quotient module M of $R\otimes_A B$ which, as a B-module, is locally free of rank d. This gives rise to a map $R\otimes_A B\to B$ given by $r\mapsto\det_M(r)$ which is homogenous of degree d. By choosing local generators of M as locally free B-module, we see that $r\mapsto\det_M(r)$ gives rise to a morphism $R\otimes_A B\to\mathbb{G}_{a,B}$ which is homogenous of degree d and therefore a point $\det(x)\in S^dV_{R/A}(B)$.

Definition 1.3.17. *We call the morphism*

$$\det_{R/A}:\mathrm{Hilb}^d_{R/A}\to S^dV_{R/A} \tag{1.8}$$

defined by the morphism of functors $x \mapsto \det(x)$ *the determinant map.*

Again, since $\mathrm{Hilb}^d_{R/A}$ is a proper scheme over A, and $S^d V_{R/A}$ is affine, the morphism $\det_{R/A}$ factors through a closed subscheme $\mathrm{Laf}^d_{R/A}$ of $S^d V_{R/A}$ which is finite over A. We thus get a proper surjective map

$$\det{}^d_L : \mathrm{Hilb}^d_{R/A} \to \mathrm{Laf}^d_{R/A}.$$

Using the nested Hilbert scheme $n\mathrm{Hilb}^d_{R/A}$ as before, we can define open subschemes $i\mathrm{Hilb}^d_{R/A}$ and $i\mathrm{Laf}^d_{R/A}$ as the complements of the images of $n\mathrm{Hilb}^d_{R/A}$. Geometric points $x \in i\mathrm{Hilb}^d_{R/A}(\bar{k})$ over $a : A \to \bar{k}$ correspond to R_a-quotient modules of R_a that are simple.

Proposition 1.3.18. *We have* $\det{}^{-1}_L(i\mathrm{Laf}^d_{R/A}) = i\mathrm{Hilb}^d_{R/A}$. *For every geometric point* $l \in i\mathrm{Hilb}^d_{R/A}(\bar{k})$ *over* $a : A \to \bar{k}$, *the group* $\mathcal{G}_{R_a}$ *acts transitively on the fiber* $(\det{}^d_L)^{-1}(l)$.

Again, this assertion is nothing but a reformulation of well known facts about modules over a finite-dimensional algebra.

Proposition 1.3.19. *Let R be a possibly non-commutative finite-dimensional algebra over a field k and M, N be R-modules which are d-dimensional k-vector spaces. Assume that* $\det_M = \det_N$ *as homogeneous polynomial of degree d on R, then M and N have isomorphic semi-simplifications. In particular, if M is a simple R-module then N is also simple and* $N \cong M$.

Proof. The assertion is obvious in one direction. If the factors M and N of R have the same semi-simplification then the induced homogenous forms $\det_M, \det_N : R \to k$ are equal because the determinant only depends on the semi-simplification. Conversely, Jacobson's density theorem implies the existence of projectors: if $V_1, \ldots, V_r$ are non-isomorphic simple R-modules, then there exists an element $e_i \in R$ which acts as identity on V_i and 0 on V_j for $j \neq i$. Now let $V_1, \ldots, V_r$ be the simple R-modules occurring as a simple subfactors of M or

N and decompose the semi-simplifications of M and N as

$$M^{ss} = V_1^{m_1} \oplus \cdots \oplus V_n^{m_r} \text{ and } N^{ss} = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r} \tag{1.9}$$

If $X_1, \ldots, X_r$ are free variables then we have the formula

$$\det_M (X_1 e_1 + \cdots + X_r e_r) = X_1^{m_1 \dim(V_1)} \ldots X_r^{m_r \dim(V_r)}$$

for the determinant of $x_1 e_1 + \cdots + x_r e_r$ on M and similarly for N. The equality

$$\det_M (X_1 e_1 + \cdots + X_r e_r) = \det_N (X_1 e_1 + \cdots + X_r e_r)$$

of polynomials of variables $X_1, \ldots, X_r$ implies that $m_i = n_i$ for all i. It follows that M and N have isomorphic semi-simplifications. □

The proof of Theorem 1.1.1 in the general case readily follows exactly as in the previous subsection by replacing the trace map by the determinant map.

Remark 1.3.20. *Notice that in this case the ring of regular functions T_R on the Lafforgue variety is not given via the simple procedure described in the introduction anymore.*

1.3.4 Dependence on the central subalgebra

If A is a commutative k-algebra contained in the center of a possibly non-commutative k-algebra R, assuming k to be algebraically closed, then Schur's lemma guarantees that A acts on every finite-dimensional simple R-module through a character $a : A \to k$. This implies that the set of k-points of iLaf doesn't depend on the choice of A. In this section, we will prove that iLaf itself is independent of the choice of a A. This will follow from a relative version of Schur's lemma.

Proposition 1.3.21. *Let R be a possibly non-commutative ring containing commutative rings $A \subset A'$ in its center. The natural morphism* $\mathrm{iLaf}_{R/A'} \to \mathrm{iLaf}_{R/A}$ *is an isomorphism.*

Proof. It is enough to prove that the morphism $i\mathrm{Hilb}_{R/A'} \to i\mathrm{Hilb}_{R/A}$ is an isomorphism. It is enough to prove that for any A-algebra B, every morphism $\mathrm{Spec}(B) \to \mathrm{iLaf}_{R/A}$ can be canonically lifted to a morphism $\mathrm{Spec}(B) \to \mathrm{iLaf}_{R/A'}$, which is the content of the following assertion. □

Proposition 1.3.22. *Let R be a possibly non-commutative ring containing commutative rings $A \subset A'$ in its center. Assume that that R is finite as an A-module. Let B be an A-algebra and M a finite locally free A-module equipped with a structure of an $(R \otimes_A B)$-module such that over every geometric point $b \in \mathrm{Spec}(B)$ over $a \in \mathrm{Spec}(A)$, M_b is a simple R_a-module. Then the ring homomorphism $A' \to End_B(M)$ factors through B.*

Proof. The homomorphism $R \to \mathrm{End}_B(M)$ is surjective as it is surjective fiberwise over $\mathrm{Spec}(B)$ by the Jacobson density theorem. It follows that the image of the central subalgebra A' is contained in B. □

1.4 Jacobson stratification and irreducibility of induced representations

Let R be a possibly non-commutative k-algebra such that there is a finitely generated subalgebra A of the center with R being finite as an A-module.

Then, by Theorem 1.1.1, the projection $\mathrm{Laf}_{R/A} \to \mathrm{Spec}(A)$ is finite which implies we can stratify $\mathrm{Spec}(A)$ according to the cardinality of the fiber of p. If $A = Z(R)$, this stratification allows us to reconstruct $\mathrm{iLaf}_{R/A}$ just from the data of the projection $p : \mathrm{Laf}_{R/A} \to \mathrm{Spec}A$. We define a notion of equivalence of algebras on the level of Lafforgue varieties that will be useful in Chapter 6. This notion of equivalence is weaker than almost Morita equivalence, but still implies a canonical bijection on the level of irreducible modules which is stronger in the sense that it preserves our geometric structure.

In the case where $\mathrm{char}(k) = 0$, A is regular, and R is a locally free A-module, we concretely describe the above stratification using the rank of the Jacobson ideal. We show that if R is a Cohen-Macaulay algebra over a Cohen-Macaulay center $Z(R)$, we can choose an appropriate regular subalgebra $A \subseteq Z(R)$ by Hironaka's miracle flatness criterion [Nag62, Theorem 25.16]. This case includes all versions of unital Hecke algebras in Section 2.

1.4.1 Equivalence of Lafforgue varieties

Let R_1, R_2 be non-commutative algebras that are finite modules over their finitely generated centers $Z_1 \coloneqq Z(R_1), Z_2 \coloneqq Z(R_2)$.

Definition 1.4.1. *We call R_1, R_2 Lafforgue equivalent if there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Laf}_{R_1/Z_1} & \xrightarrow{\cong} & \mathrm{Laf}_{R_2/Z_2} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(Z_1) & \xrightarrow{\cong} & \mathrm{Spec}(Z_2) \end{array}$$

where the vertical arrows are the projections of Theorem 1.1.1 and the horizontal arrows are isomorphisms.

We denote Lafforgue equivalence by $R_1 \overset{L}{\sim} R_2$.

Proposition 1.4.2. *If R_1, R_2 are almost Morita equivalent they are also Lafforgue equivalent, ie.*

$$R_1 \overset{a}{\sim} R_2 \implies R_1 \overset{L}{\sim} R_2.$$

Proof. Assume $R_1 \overset{a}{\sim} R_2$. Then, for $Z_i = Z(R_i)$, we have that $R_1 \overset{a}{\sim} R_2 \implies Z_1 \cong Z_2$.

Without loss of generality, assume $A = Z_1 = Z_2$. Then, for any commutative A-algebra B, we have that $R_1 \overset{a}{\sim} R_2 \implies R_1 \otimes_A B \overset{a}{\sim} R_2 \otimes_A B$.

Therefore,

$$\mathrm{Hilb}_{R_1/A}(B) \cong \mathrm{Hilb}_{R_2/A}(B)$$
$$n\mathrm{Hilb}_{R_1/A}(B) \cong n\mathrm{Hilb}_{R_2/A}(B)$$

since the definition of the (nested) non-commutative Hilbert scheme involves only finite-dimensional modules. □

Proposition 1.4.3. *If $R_1 \overset{L}{\sim} R_2$ then if we denote by $f : \mathrm{Laf}_{R_1/Z_1} \to \mathrm{Laf}_{R_2/Z_2}$ the upper horizontal isomorphism in the diagram*

$$\begin{array}{ccc} \mathrm{Laf}_{R_1/Z_1} & \xrightarrow{\cong} & \mathrm{Laf}_{R_2/Z_2} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(Z_1) & \xrightarrow{\cong} & \mathrm{Spec}(Z_2) \end{array}$$

then f restricts to an isomorphism $\bar{f} : \mathrm{iLaf}_{R_1/Z_1} \xrightarrow{\cong} \mathrm{iLaf}_{R_2/Z_2}$.

Proof. Consider the function $f : \mathrm{Laf}_{R_i/Z_i}(\mathbb{C}) \to \mathbb{Z}$, defined by $f(x) = |p^{-1}(p(x))|$. Then, iLaf_{R_i/Z_i} is the maximal open set such that f is continuous. Since iLaf_{R_i/Z_i} can be determined only from p, the result follows. □

1.4.2 Jacobson stratification

Let A be a commutative ring contained in the center of a possibly non-commutative ring R. From now on, we will assume that R is a finite locally free A-module. We will also assume that A contains a field k of characteristic zero.

For every point $a : A \to k(a)$ of $\mathrm{Spec}(A)$, $k(a)$ being a field, the fibre $R_a = R \otimes_A k(a)$ is a finite-dimensional $k(a)$-algebra. The Jacobson radical $J_a = \mathrm{rad}(R_a)$, defined as the intersection of all maximal left ideals of R_a, is a 2-sided ideal which can be characterized in multiple ways, namely, it is the intersection of the annihilators of simple left R_a-modules, or the maximal left (or right) nilpotent ideals, see [Lam91, 4.2,4.12]. The quotient R_a/J_a is a semi-simple k-algebra which, by the Artin-Weddenburn theorem, is isomorphic to a product of matrix algebras $R_a/J_a = \prod_{i=1}^r M_{n_i}(D_i)$ where $M_{n_i}(D_i)$ is a matrix algebra over a skew field D_i containing $k(a)$ in its center.

Proposition 1.4.4. *The function* $r_{\mathrm{Jac}} : \mathrm{Spec}(A)$ *given by* $a \mapsto \dim_{k(a)} J_a$ *is upper semi-continuous.*

The assertion will follow from yet other interpretation of the Jacobson radical as the kernel of a trace form. We recall that as R is a finite locally free A-module, for every element $r \in R$, the A-linear operator on R given by $x \mapsto rx$ has a well defined trace $\mathrm{tr}_{R/A}(r)$. It follows that we have a symmetric A-bilinear form on R given by $Tr_{R/A}(x,y) = \mathrm{tr}_{R/A}(xy)$, or equivalently a A-linear map $Tr_{R/A} : R \to R^\vee$. The construction of the trace form and the bilinear form $Tr_{R/A}$ commute in the obvious way with base change and for every geometric point $a : A \to k(a)$, we have a trace form $\mathrm{tr}_a : R_a \to k(a)$ and a symmetric bilinear form $Tr_{R/A,a}$ on R_a, or equivalently a linear form $Tr_{R/A,a} : R_a \to R_a^\vee$.

Proposition 1.4.5. *For every point* $a : A \to k(a)$ *of* $\mathrm{Spec}(A)$*, the Jacobson radical* J_a *is the kernel the bilinear form* $Tr_{R/A,a} : R_a \to R_a^\vee$.

Proof. Since J_a is a nilpotent ideal, for every $x \in J_a$ and $y \in R_a$, we have $\mathrm{tr}_a(xy) = 0$. It follows that J_a is contained in the kernel of $Tr_{R/A,a}$. Moreover, the Artin-Weddenburn theorem implies that $Tr_{R/A,a}$ induces a non-degenerate bilinear form on R_a/J_a and therefore J_a is exactly equal to the kernel of $Tr_{R/A,a}$. □

We will now construct the stratification of $\mathrm{Spec}(A)$ by the rank of the Jacobson ideal using the concept of determinantal ideals. Assume that R is a locally free A-module of rank n. Locally for the Zariski topology we may assume that R is a free A-module of rank n, and the trace form $Tr : R \to R^\vee$ is given by a $n \times n$-matrix. For every positive integer i, we define I_i to be the ideal of A such that locally for the Zariski topology, I is generated by the minors to the order $n-i+1$ of the local matrix of Tr. We know a chain of inclusions of ideals $0 = I_0 \subset I_1 \subset \cdots$ which induces a chain of inclusion of closed subsets $\bar{X}_0 \supset \bar{X}_1 \supset \cdots$ where $\bar{X}_i = \mathrm{Spec}(A/I_i)$. Over X_i the complement of $\bar{X}_{i+1}$ in $\bar{X}_i$, the rank of the Jacobson radical is constant of value i.

In fact, over X_i the trace form $Tr_{X_i} : R \otimes_A \mathcal{O}_{X_i} \to R^\vee \otimes_A \mathcal{O}_{X_i}$ has kernel a locally free $\mathcal{O}_{X_i}$-module J_i of rank i, and image a locally free $\mathcal{O}_{X_i}$-module $\bar{R}_{X_i}$ of rank $n-i$. The trace form Tr_{X_i} induces a non-degenerate symmetric bilinear form $\bar{T}r_{X_i}$ on $\bar{R}_{X_i}$. In particular, for every point $a : A \to k(a)$ of $\mathrm{Spec}(A)$ belonging to the stratum X_i, $\bar{R}_{X_i} \otimes_{\mathcal{O}_{X_i}} k(a)$ is a semi-simple algebra over $k(a)$. Let $\bar{a} : A \to \overline{k(a)}$ a geometric point over a. Then $\bar{R}_{X_i} \otimes_{\mathcal{O}_{X_i}} \overline{k(a)}$ is isomorphic to a product of matrix algebras

$$\bar{R}_{X_i} \otimes_{\mathcal{O}_{X_i}} \overline{k(a)} = \prod_{i=1}^{r} M_{n_i}(\overline{k(a)})$$

where $\underline{n}(a) = (n_1, \ldots, n_r)$ is unordered sequence of positive integers depending only on a.

Proposition 1.4.6. *The function $a \mapsto \underline{n}(a)$ is locally constant on X_i.*

Proof. Since $\bar{R}_{X_i}$ is a locally free $\mathcal{O}_{X_i}$-module equipped with a structure of an associative algebra which is fiberwise semisimple over X_i, its invertible elements define a smooth group scheme $\mathcal{G}_{\bar{R}_{X_i}}$ over X_i. Its geometric fiber over a geometric point $\bar{a}$ is isomorphic to $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. Thus $\mathcal{G}_{\bar{R}_{X_i}}$ is a smooth reductive group scheme whose geometric fiber over $\bar{a}$ is isomorphic to $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$. A general theorem in SGA 3 on smooth reductive group schemes implies that the function $a \mapsto \underline{n}(a)$ is locally constant [ABD+66, Exposé XIX, Corollaire 2.6]. □

1.4.3 Cohen-Macaulay property

For a (twisted) affine Hecke algebra, we can choose an appropriate regular subalgebra to apply the Jacobson stratification by using its explicit presentation. In this subsection, we show that the existence of such a regular subalgebra is guaranteed by a condition that appears more often in the literature.

Proposition 1.4.7. *Let R be a possibly non-commutative k-algebra and $Z(R)$ its center. Assume $Z(R)$ is finitely generated and R is a finite $Z(R)$-module. The following properties are equivalent*

1. *R is a finite Cohen-Macaulay module over its center $Z(R)$. $Z(R)$ is a Cohen-Macaulay k-algebra.*

2. *R is a finite free A-module for a finitely generated regular central subalgebra $A \subseteq Z(R)$.*

If $Z(R)$ is regular, we can take $A = Z(R)$.

Proof. (1) $\implies$ (2) : Since $Z(R)$ is finitely generated, by Noether normalization [EE95, Theorem 13.3] we have a regular subalgebra $A \subseteq Z(R)$ such that $Z(R)$ is a finite A-module and therefore R is also a finite A-module. For R local, the proposition is Hironaka's miracle flatness criterion [Nag62, Theorem 25.16]. For A regular, a locally free module is free [Lin82, Theorem].

(2) $\implies$ (1) : For R local, the proposition is again Hironaka's miracle flatness. Cohen-Macaulayness is a local property. □

Example 1.4.8. *The Hecke algebra $\mathcal{H}_{\mathfrak{s}}(G)$ of a Bernstein component $\mathfrak{s} \in \mathfrak{B}(G)$ satisfies condition (1) [BBK18, Proposition 3.1].*

1.5 Generalized discriminants

Let R be a possibly non-commutative k-algebra that is a free finite module over a finitely generated subalgebra A of its center $Z(R)$. We assume k to be an algebraically closed field of characteristic 0.

In this section, we focus on the open dense stratum $X_0 \subseteq \mathrm{Spec}A$. If the Jacobson radical of R is trivial, X_0 is the semisimplicity locus of R. We characterize X_0 as the zero set of a generalized discriminant ideal.

For the Hecke algebra $\mathcal{H}_{\mathfrak{s}}$ of a Bernstein component $\mathfrak{s} \in \mathfrak{B}(G)$. Then, the projection $\mathrm{Laf}_{\mathcal{H}_{\mathfrak{s}}/Z_{\mathfrak{s}}} \to \mathrm{Spec}(Z_{\mathfrak{s}})$ sends a smooth irreducible representation ρ to its cuspidal support $\mathbf{sc}(\rho) \coloneqq (M,\sigma)$, defined by $\rho \hookrightarrow i_M^G(\sigma)$. For a generic (M,σ), the induced representation $i_M^G(\sigma)$ is irreducible. As a main application of the results in this section, we prove Theorem 1.1.2 providing a computational criterion for the irreducibility of $i_M^G(\sigma)$ outside a singular locus.

We provide computational tools for the discriminant in cases where there is an explicit presentation as in the case of (twisted) affine Hecke algebras. In particular, we compute the discriminant for the Iwahori-Hecke algebra of a split reductive p-adic group, first for the case of an adjoint group where the center is already regular, and then for the general case, where we need to choose a regular subalgebra.

1.5.1 Definition and properties

If $f : R \to R'$ is a map of free A-modules of rank n, the n-th exterior power $\bigwedge^n f : \bigwedge^n R \to \bigwedge^n R'$ is a map of free A-modules of rank 1 [Bou89, Chapter 7, Theorem 8.1]. In the case $R' = R$, by means of the canonical isomorphism $\mathrm{End}_A(A) \cong A$ given by the inverse homomorphisms $a \to r_a(x) = ax$ and $r \to r(1)$, we can canonically associate to $f : R \to R$ its determinant $\det(f)$.

From now on we assume R is also an A-algebra.

Definition 1.5.1. *The norm function $N_{R/A} : End_A(R) \to A$ is the map sending an endomorphism $f \in End_A(R)$ to $N_{R/A}(f) \coloneqq \det(f)$. If $r \in R$, by associating to r the endomorphism $f_r \in End_A(R)$ given by $f_r(x) = rx$, we also define $N_{R/A}(r) \coloneqq N_{R/A}(f_r)$.*

We recall some elementary properties of the norm.

Lemma 1.5.2. *For the norm function $N_{R/A}$ we have*

- $N_{R/A}(fg) = N_{R/A}(f)N_{R/A}(g)$,
- *If $a \in A$, $N_{R/A}(a) = a^n$ where n is the rank of R over A.*

Proof. The first part follows from multiplicativity of the determinant. For the second, a can be identified with a scalar matrix. □

The next Lemma is less trivial than it may appear, for a proof, see [Cas86, Appendix B, Lemma 4].

Lemma 1.5.3. *If B is a commutative A-algebra that is free as an A-module and C is a B-algebra such that C is locally free over B, we have*

$$N_{C/A} = N_{B/A} \circ N_{C/B}.$$

In particular, if n is the rank of C over B,

$$N_{C/A}(b) = \left(N_{B/A}(b)\right)^n$$

When $R' \neq R$, we can only identify $\bigwedge^n f$ with an element of A after choosing bases. Nonetheless, if $R' = R^\vee$, a basis $b = \{r_1, \ldots, r_n\}$ of R as an A-module uniquely determines a dual basis $b^\vee$ of $R^\vee := \mathrm{Hom}_A(R, A)$. By the universal property of free modules, we now have canonical identifications

$$\begin{aligned} \bigwedge^n R &\cong A \\ \bigwedge^n R^\vee &\cong A \\ \left(\bigwedge^n f\right)_b \in \mathrm{End}_A(A) &\cong A \end{aligned}$$

If $\bar{b} = Mb$ is another basis for R, then $\det(M)$ is an invertible element of A, and the dual basis is given by $\bar{b}^\vee := {}^tMb^\vee$. Thus,

$$\left(\bigwedge^n f\right)_{\bar{b}} = \det(M) \cdot \left(\bigwedge^n f\right)_b \cdot \det({}^tM) = \det(M)^2 \cdot \left(\bigwedge^n f\right)_b$$

Definition 1.5.4. *Let $Tr_{R/A} : R \otimes_A R \to A$ be the trace form defined by $Tr_{R/A}(x,y) = tr_A(xy)$. We consider it as a function $Tr_{R/A} : R \to R^\vee$. By the preceding paragraph, all possible choice of a basis b for R define the same element $\left(\bigwedge^n Tr_{R/A}\right)_b$ up to multiplication by $A^\times$, and therefore generate the same principal ideal $d_{R/A}$.*

We call $d_{R/A}$ the discriminant of R over A.

Remark 1.5.5. *Any choice of a generator for $d_{R/A}$ provides us with the same regular function on* $\mathrm{Spec}(A)$*, so we often treat $d_{R/A}$ as a function.*

Remark 1.5.6. *In the case of number rings, Definition 1.5.4 agrees with the classical discriminant of algebraic number theory.*

As we mainly use the Jacobson stratification over the open dense stratum X_0, our definition is motivated by the following lemma.

Lemma 1.5.7. *The open stratum X_0 in the Jacobson stratification of* $\mathrm{Spec}A$ *is the complement of the zero set $V(d_{R/A})$*

Proof. Notice that the zero set is well-defined since any two elements of the discriminant are related by an invertible element, thus the zero set does not vary. By Definition 1.5.4, the zero set is the locus where the trace form is an isomorphism, thus the Jacobson radical is trivial by Proposition 1.4.5. □

In the case of number fields, we have an elementary formula allowing us to compute the discriminant of a tower of extensions in terms of the discriminants of the intermediate steps. As stated in the introduction, it turns out it can be generalized to our case. We recall Lemma 1.1.3.

Lemma 1.1.3. *For a tower of extensions $C/B/A$ such that A, B are commutative and regular, C is commutative, C is free of rank n as a B-module and B is free as an A-module, we have that*

$$d_{C/A} = (d_{B/A})^n \cdot N_{B/A}(d_{C/B}),$$

where $N_{B/A}$ is the norm function.

Proof. Let $X = \mathrm{Spec}A, Y = \mathrm{Spec}B, Z = \mathrm{Spec}C$ and $g : Z \to Y$ and $f : Y \to X$ the maps corresponding to inclusion.

Let $R_{Y/X}$ be the ramification divisor for f. By reducing to the local case, $f_* R_{Y/X} = div(d_{B/A})$. We use the relative short exact sequence of Kahler differentials, where injectivity follows by the fact that all maps are smooth

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

We take determinants in the sense of [Har77, Exercise II.6.11], to get

$$\det(\Omega_{C/A}) \cong \det(\Omega_{B/A} \otimes_B C) \otimes \det(\Omega_{C/B}).$$

Now by the smoothness of f, g we have $\det(\Omega_{C/A}) = \omega_{C/A}$, $\det(\Omega_{B/A}) = \omega_{C/B}$ and thus $\det(\Omega_{B/A} \otimes_B C) = g^* \omega_{B/A}$. Therefore,

$$\omega_{C/A} \cong \omega_{C/B} \otimes g^* \omega_{B/A}.$$

We know that $\omega_{C/A} = \mathcal{L}(R_{Z/X})$ is the invertible sheaf corresponding to the ramification divisor $R_{Z/X}$. Thus, taking associated divisors,

$$R_{Z/X} \cong R_{Z/Y} + g^* R_{Y/X}$$

We consider the pushforward by $f \circ g$ to get the divisor corresponding to the discriminant. Since $g_* g^*$ for divisors is multiplication by the degree, and $f_* div(z) = div(N(z))$, we get

$$\begin{aligned} div(d_{C/A}) &\cong f_* div(d_{C/B}) + f_*(nR_{Y/X}) \\ &\cong div(N_{B/A}(d_{C/B})) + div((d_{B/A})^n) \\ &\cong div((d_{B/A})^n N_{B/A}(d_{C/B})) \end{aligned}$$

□

Remark 1.5.8. *We can also deduce Lemma 1.1.3 by repeated application of the generalized Riemann-Hurwitz formula.*

Indeed, we have

$$\begin{aligned} R_{Z/X} &\cong K_Z - (f \circ g)^* K_X \\ &\cong K_Z - g^* f^* K_X \\ &\cong K_Z - g^*(K_Y - R_{Y/X}) \\ &\cong R_{Z/Y} + g^* R_{Y/X} \end{aligned}$$

and we conclude as before.

1.5.2 Irreducibility of induced representations

In the case of the Hecke algebra $\mathcal{H}_{\mathfrak{s}}$ of a Bernstein component $\mathfrak{s} \in \mathfrak{B}(G)$, we know after [BBK18, Proposition 3.1] that $\mathcal{H}_{\mathfrak{s}}$ is a finite Cohen-Macaulay module over its center $Z_{\mathfrak{s}}$ which is itself a Cohen-Macaulay algebra. We can apply the Lafforgue variety construction for $R = \mathcal{H}_{\mathfrak{s}}$ and $A = Z_{\mathfrak{s}}$, but using Proposition 1.4.7 we can also apply it for A being a regular algebra contained in $Z_{\mathfrak{s}}$ such that $Z_{\mathfrak{s}}$ is a finite A-module. If A is regular, then both $\mathcal{H}_{\mathfrak{s}}$ and $Z_{\mathfrak{s}}$ are finite locally free A-modules. In this case, the group scheme $\mathcal{G}_{R/A}$ is smooth acting on the Hilbert scheme $\mathrm{Hilb}_{R/A}$, which is a closed subscheme of $\mathcal{A}_{R/A}$, the familiar relative Grassmannian scheme attached to a vector bundle.

If $f : \mathrm{Spec}(Z_{\mathfrak{s}}) \to \mathrm{Spec}(A)$ is the projection corresponding to the inclusion $A \subseteq Z_{\mathfrak{s}}$, we define $Z(f)$ to be the closed subset of $\mathrm{Spec}(Z_{\mathfrak{s}})$ where f is not smooth. Let X_0 be the open dense stratum of $\mathrm{Spec}(Z_{\mathfrak{s}})$ given by the cardinality of the fiber of the projection from the Lafforgue variety. We identify a cuspidal datum (M,σ) with the corresponding point in $\mathrm{Spec}(Z_{\mathfrak{s}})$.

We recall and prove Theorem 1.1.2.

Theorem 1.1.2. *Let $(M,\sigma)_G$ be a cuspidal datum. Then, $i_M^G(\sigma)$ is irreducible if and only if $(M,\sigma) \in X_0$. Outside of the singular locus $Z(f)$, this is equivalent to*

$$d_{\mathcal{H}_{\mathfrak{s}}/A}(f(M,\sigma)) \neq 0.$$

Proof. By Theorem 1.1.1, we get finite projections

$$\mathrm{Laf}_{\mathcal{H}_{\mathfrak{s}}/Z_{\mathfrak{s}}} \cong \mathrm{Laf}_{\mathcal{H}_{\mathfrak{s}}/A} \xrightarrow{p} \mathrm{Spec}(Z_{\mathfrak{s}}) \xrightarrow{f} \mathrm{Spec}(A)$$

By the definition of p, $|JH(i_M^G(\sigma))| = |p^{-1}(M,\sigma) \cap \mathrm{iLaf}_{\mathcal{H}_{\mathfrak{s}}/A}|$. Since generically an induced representation is irreducible, over X_0 the cardinality of the fiber is 1 which proves the first assertion.

Let Y_0 be the open dense stratum of the Jacobson stratification for $\mathrm{Spec}(A)$, and $n = deg(f)$. Then, for a generic point $a \in \mathrm{Spec}(A)$, the cardinality of $f \circ p$ is n, and thus the cardinality of a point $a \in \mathrm{Spec}(A)$ is $\geq n$ with equality if and only if $a \in Y_0$.

Since the fibers of f outside the singular locus have cardinality n, Lemma 1.5.7 implies the second assertion. □

Example 1.5.9. *Let H be the Iwahori-Hecke algebra of $\mathrm{GL}_2(F)$. By Example 1.2.7, the Weyl group is $W = S_2$. By Proposition 1.2.7, H is generated over the group algebra of the cocharacter lattice $R \cong \mathbb{C}[x_1^{\pm}, x_2^{\pm}]$ by $T_e = 1$ and an element T_s satisfying*

$$(T_s + 1)(T_s - q) = 0 \tag{1.10}$$

$$T_s x_1 = x_2 T_s + (q-1)x_1 \tag{1.11}$$

The center is $R^W = \mathbb{C}[x_1^{\pm}, x_2^{\pm}]^{S_2}$. A cuspidal datum in this case corresponds to a choice of an unordered pair of complex numbers defining an unramified character of a split maximal torus.

Let V be a simple H-module. The subalgebra R is abelian therefore we can choose a common eigenvector $v \in V$. By equation 1.11, every element $h \in H$ can be written $h = T_e r_1 + T_s r_2$ for $r_1, r_2 \in R$. Since v is cyclic by simplicity of V, $\dim(V) \leq 2$. By equation 1.10, if $\dim(V) = 2$, then $tr_{T_s}(V) = q - 1$, and if $\dim(V) = 1$ then T_s acts as either -1 or q.

Therefore, the trace ring is $T_H = \mathbb{C}[x_1^{\pm}, x_2^{\pm}]^{S_2} \oplus \mathbb{C}[x_1^{\pm}] \oplus \mathbb{C}[x_1^{\pm}]$. The Lafforgue variety and the projection are therefore roughly given by the following picture.

In this case, the center R^W is already regular, and the discriminant is

$$d_{H/R^W} = (x_2 - qx_1)^2 (x_1 - qx_2)^2,$$

which retrieves that the induction $i_M^G(\chi_1, \chi_2)$ is irreducible if and only if $\chi_1 \chi_2^{-1} \neq q^{\pm}$. When

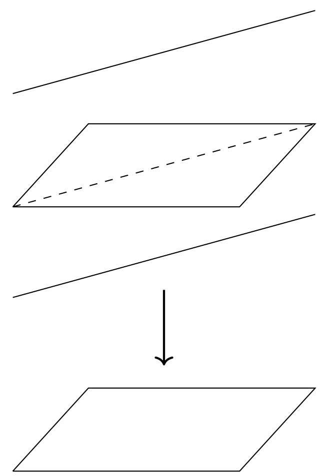

Figure 1.1: Projection from the Lafforgue variety to the Bernstein variety for Iwahori representations of $GL_2(F)$

this is not the case, the Jordan-Holder constituents of the induction are an irreducible character and a Steinberg representation corresponding to the two other connected components shown in Figure 1.1.

1.5.3 Discriminant of adjoint reductive groups

The center R^W of the Iwahori-Hecke algebra is also the coordinate ring of $\hat{T} /\!\!/ W$ where $\hat{T}$ is the dual torus. In this subsection we assume G is an adjoint group, thus $\hat{T}$ is the torus of a simply-connected group, and in this case $\hat{T} /\!\!/ W \cong \mathbb{A}^r$ where r is the rank of G. Thus, for an adjoint group R^W is regular. In this case, we can retrieve Kato's result by computing d_{H/R^W}.

The computation essentially will be performed in two steps, from H to R and from R to R^W, in a similar fashion to Lemma 1.1.3, which cannot be used directly since H is non-commutative. It turns out that the discriminant behaves in a similar way nonetheless.

Let $W = \{w_1, \ldots, w_n\}$ and I_{w_i}, K_{w_i} be the intertwiners/ normalized intertwiners as we defined them in Section 2. Then, d_{H/R^W} is the discriminant of the lattice $\{I_{w_i}\pi^{\mu_j}\}_{i,j}$ for proper μ_j.

Thus, we need to compute the determinant

$$\det(\{tr(I_{w_i}\pi^{\mu_j}I_{w_k}\pi^{\mu_l})\}_{i,j,k,l\in[n]}) = \det(\{tr(I_{w_i}I_{w_k}\pi^{w_k(\mu_j)}\pi^{\mu_l})\}_{i,j,k,l\in[n]}).$$

We notice that for $w_i \neq w_k^{-1}$ the trace is zero, because elements I_w for $w \neq e$ permute the generelized eigenvectors, so we have n $n\times n$ blocks. Also, we recall that setting $e_a = 1 - q^{-1}\pi^{a^\vee}, d_a = 1 - \pi^{a^\vee}$, gives

$$I_w I_{w^{-1}} = \prod_{a\in R_w} \frac{e_a e_{-a}}{d_a d_{-a}}$$

by Lemma 1.2.35.

Thus, we can simplify the calculation using the following.

Lemma 1.5.10. *Let R be a commutative algebra over the commutative algebra A. Let $p, r_1, \ldots, r_n \in R$. Then we have that*

$$\det(\{tr(pr_ir_j)\}) = N_{R/A}(p)\cdot\det(\{tr(r_ir_j)\}).$$

Proof. Consider a basis of generalized eigenvectors v_i and let κ_i be the eigenvalues of p and λ_i^j be the eigenvalues of r_j. Then $tr(pr_ir_j) = \sum \kappa_k\lambda_k^i\lambda_k^j$. We therefore have

$$\begin{pmatrix} tr(pr_1^2) & \ldots & tr(pr_1r_n) \\ \vdots & \ddots & \vdots \\ tr(pr_nr_1) & \ldots & tr(pr_n^2) \end{pmatrix} = \begin{pmatrix} \kappa_1\lambda_1^1 & \ldots & \kappa_1\lambda_1^n \\ \vdots & \ddots & \vdots \\ \kappa_n\lambda_n^1 & \ldots & \kappa_n\lambda_n^n \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^1 & \ldots & \lambda_n^1 \\ \vdots & \ddots & \vdots \\ \lambda_1^n & \ldots & \lambda_n^n \end{pmatrix}$$

The above product is equal to

$$\kappa_1\cdots\kappa_n\cdot\begin{pmatrix} \lambda_1^1 & \ldots & \lambda_1^n \\ \vdots & \ddots & \vdots \\ \lambda_n^1 & \ldots & \lambda_n^n \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^1 & \ldots & \lambda_n^1 \\ \vdots & \ddots & \vdots \\ \lambda_1^n & \ldots & \lambda_n^n \end{pmatrix} = \det(p)\cdot\begin{pmatrix} tr(r_1^2) & \ldots & tr(r_1r_n) \\ \vdots & \ddots & \vdots \\ tr(r_nr_1) & \ldots & tr(r_n^2) \end{pmatrix}$$

$\square$

By Lemma 1.5.10, we get

$$d_{H/R^W} = \left(\prod_{i=1}^{n} \prod_{a \in R_{w_i}} N_{R/R^W} \left(\frac{e_a e_{-a}}{d_a d_{-a}} \right) \right) \cdot d^n_{R/R^W}$$

Notice that we also have

$$\prod_{i=1}^{n} \prod_{a \in R_{w_i}} \frac{e_a e_{-a}}{d_a d_{-a}} = \prod_{a \in \Phi} \prod_{w \in W, a \in R_w} \frac{e_a e_{-a}}{d_a d_{-a}} = \left(\prod_{a \in \Phi} \frac{e_a e_{-a}}{d_a d_{-a}} \right)^{n/2},$$

so, since this element is W-invariant and thus by Lemma 1.5.2 its norm is itself to the n-th power, we get the general formula

$$d_{H/R^W} = \left(\prod_{a \in \Phi} \frac{e_a e_{-a}}{d_a d_{-a}} \right)^{n^2/2} \cdot d^n_{R/R^W}. \tag{1.12}$$

We can compute the discriminant for R/R^W. Indeed, it is enough to calculate the ramification divisor of the map $\mathrm{Spec} R \to \mathrm{Spec} R^W$. Ramification happens when $d_a = 0$ for some a. Indeed, in that case, the corresponding homomorphism $R \to k$ was s_a-invariant, and in that case two sheets degenerate in one in every ramified point. Thus, d_a appears with an exponent of 1 in the ramification divisor. Pushed forward, we have that the discriminant of the extension R/R^W is $(\prod_{a \in \Phi} d_a)^n$. This computation is also carried out algebraically by Steinberg [Ste74, pp. 125-127].

Combined with the fact that $d_a = d_{-a}$ up to an invertible element, (1.12) shows that for an adjoint group G we have

Proposition 1.5.11. *If G is adjoint, we have*

$$d_{H/R^W} = \left(\prod_{a \in \Phi} e_a e_{-a} \right)^{n^2/2}.$$

Remark 1.5.12. *Since the zero locus of* d_{H/R^W} *is exactly the locus where the induced representation is reducible by the considerations in Section 3, for the case of an adjoint group we retrieve Kato's result [Kat81, Theorem 2.2]. Notice that for an adjoint group the second condition of Kato's theorem is always true.*

1.5.4 Discriminant in the non-adjoint case

If G is not adjoint, R^W is not regular anymore, so we need to restrict to some subalgebra A that is regular. We make a canonical choice.

Definition 1.5.13. *We identify the fundamental weights* $\omega_1, \dots, \omega_n$ *with the trace function of the corresponding fundamental representations. Then, we consider the smallest integers* $d_1, \dots, d_n$ *such that* $\omega_i^{d_i} \in R^W$. *We define* $A = \mathbb{C}[\omega_1^{d_1}, \dots, \omega_n^{d_n}] \subseteq R^W$ *to be the algebra of fundamental weights.*

A is obviously regular as it is a polynomial algebra.

By the same procedure as in the previous subsection, equation 1.12 is still true upon replacing d_{R/R^W} by $d_{R/A}$, so we want to compute $d_{R/A}$. Recall that R is the group algebra of the cocharacter lattice, so alternatively, it is the function ring of the dual torus $\hat{T}$. The fact that R^W is regular when G is adjoint comes from the fact that the dual torus would be simply connected: indeed, in that case it is known that $\hat{T} /\!\!/ W$ is an affine space of dimension the rank of G, and it is given as polynomials over the trace functions corresponding to the fundamental weights.

For the general case, we consider a simply connected cover $\tilde{T}$ of $\hat{T}$ such that $\hat{T} = \tilde{T} /\!\!/ Z$. We define $R^+ = k[\tilde{T}]$. Then R^+ is regular and $(R^+)^W$ is also regular.

Consider the following diagram

$$\begin{tikzcd}
 & R \arrow[r, dash] \arrow[d, dash] & R^+ \arrow[d, dash] \\
 & R^W \arrow[r, dash] \arrow[dl, dash] & (R^+)^W \\
A & &
\end{tikzcd}$$

We want to compute $d_{R/A}$. By Lemma 1.1.3, it is enough to compute the discriminants

$$d_{R^+/R}, d_{R^+/(R^+)^W}, d_{(R^+)^W/A}.$$

We already know $d_{R^+/(R^+)^W}$. Since $\hat{T} /\!\!/ W \cong \mathbb{C}[\omega_1, \dots, \omega_n]$, we have

$$d_{(R^+)^W/A} \cong \omega_1^{d_1(d_1-1)} \cdots \omega_n^{d_n(d_n-1)}.$$

We also have that $N_{R/A}(d_{R^+/R})$ is an invertible element. Let $n = |W|$ as before, and $[R^+ : R] = r$. Then if we set $[R : A] = nd$ we have $[(R^+) : A] = rd$. By using Lemma 1.1.3 we get the following theorem.

Proposition 1.5.14. *In the general case, and for A being the algebra of fundamental weights, we have*

$$d_{H/A} = \left(\prod_{a\in\Phi} e_a e_{-a}\right)^{dn^2/2} \cdot \left(\omega_1^{d_1(d_1-1)} \cdots \omega_n^{d_n(d_n-1)}\right)^{n/r}.$$

Proof. By the same method as in the adjoint case,

$$d_{H/A} = \left(\prod_{a\in\Phi} \frac{e_a e_{-a}}{d_a d_{-a}}\right)^{dn^2/2} \cdot d_{R/A}^n.$$

By Lemma 1.1.3, writing the discriminant $d_{R^+/A}$ in two different ways and ignoring the invertible factor $N_{R/A}(d_{R^+/R})$ we have

$$d^r_{R/A} = (d_{R^+/(R^+)^W})^{rd} \cdot d^n_{(R^+)^W/A}.$$

Combining the two equations with $d_{R^+/(R^+)^W} = (\prod_{a\in\Phi} d_a)^n$ gives the result. □

Example 1.5.15 (SL_2 case)**.** *For SL_2 the dual group is PGL_2, and the simply connected cover would be again SL_2. This gives $R^+ = \mathbb{C}[x^{\pm}]$, while $R^+ /\!\!/ \mathbb{Z}_2 = \mathbb{C}[x^{\pm 2}]$. Then $(R^+)^W \cong \mathbb{C}[x + x^{-1}]$ and $R^W = \mathbb{C}[x^2 + x^{-2}] = A$ - this is the only simply connected group for which that is correct, since $\omega_1 = x + x^{-1}$ so ω_1^2 generates R^W.*

It is easy now to compute directly $d_{R^+/R} = x^2, d_{R^+/(R^+)^W} = (1 - x^{-2})^2, d_{(R^+)^W/A} = (1 + x^{-2})^2$, which gives (one can also do this directly to get the same result) $d_{R/A} = (1 - x^{-4})^2$ since x^2 is invertible.

Therefore, either by Proposition 1.5.14 or by direct computation,

$$d_{H/R^W} = (1 - q^{-1}\pi^{a^\vee})^2 \cdot (1 - q^{-1}\pi^{-a^\vee})^2 \cdot (1 + \pi^{a^\vee})^4.$$

As a corollary, the induced representation is irreducible if and only if one of the three factors is zero. The same result can be obtained from Kato's theorem or a direct calculation of the conditions [Sol21, pp. 1020].

1.6 Application to the Local Langlands Conjecture

Let F be a non-archimedean local field with ring of integers $\mathcal{O}$ and uniformizer τ. Let $k = \mathcal{O}/\tau\mathcal{O}$ be the residue field and $q = |k|$. Set $G := \mathbf{G}(F)$ to be the group of F-points of a connected reductive algebraic group $\mathbf{G}$. Every connected reductive group is the inner twist of a quasi-split group [Spr, §16.4].

Let W_F be the Weil group of F and I_F the inertia group. We denote by $W'_F := W_F \times SL_2(\mathbb{C})$ the Weil-Deligne group.

Over a separable closure F_{sep} of F, the group $\mathbf{G}$ is split, therefore by the isogeny theorem it is classified by its root data, see Section 2. By the symmetry in the definition of a root datum, we can define the *Langlands dual group* $\mathbf{G}^\vee$ which corresponds via the isogeny theorem to the dual root datum. We denote by $G^\vee := \mathbf{G}^\vee(\mathbb{C})$ the group of $\mathbb{C}$-points. The group W_F acts via its Galois action on $\mathbf{G}$ and $\mathbf{G}^\vee$.

Definition 1.6.1. *Let G be a reductive group over F. We define ${}^L G := G^\vee \rtimes W_F$ to be the L-group of G.*

Remark 1.6.2. *If G is F-split, the action of W_F is trivial, therefore we have ${}^L G := G^\vee \times W_F$.*

Definition 1.6.3. *A continuous group homomorphism $\phi : W_F' \to {}^L G$ is called an L-parameter for G if*

1. *$\phi(w) \in {}^L G w$ for all $w \in W_F$*
2. *$\phi(w)$ is semisimple for all $w \in W_F$*
3. *$\phi|_{SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to {}^L G$ is a homomorphism of algebraic groups.*

The Local Langlands conjecture asserts the existence of a surjective, finite-to-one map

$$LLC : \operatorname{Irr}(G) \to \Phi(G)$$

from the set of smooth irreducible representations of G to the set $\Phi(G)$ of $G^\vee$-conjugacy classes of L-parameters for G.

This correspondence has been refined to a bijection with a set $\Phi_e(G)$ of *enhanced L-parameters*. Corresponding to the Bernstein decomposition on the group side there is a similar partition of $\Phi_e(G)$ on the Galois side in sets $\Phi_e^{\mathfrak{s}^\vee}(G)$.

In [Sol22], Solleveld proves the existence of an extended affine Hecke algebra $H_{\mathfrak{s}}$ such that $\operatorname{Irr}^{\mathfrak{s}}(G) \cong \operatorname{Irr}(H_{\mathfrak{s}})$. In [AMS21], Aubert, Moussaoui and Solleveld construct an affine

Hecke algebra $H_{\mathfrak{s}^\vee}$ such that $\Phi_e^{\mathfrak{s}^\vee}(G) \cong \mathrm{Irr}(H_{\mathfrak{s}^\vee})$. Many cases of the Local Langlands Correspondence have been established in this way by comparing the two Hecke algebras [AMS22].

In this section, we propose to compare the two Lafforgue varieties given by the Hecke algebras of both sides.

Conjecture 1.6.4. *There is a commutative diagram*

$$\begin{array}{ccc} Laf_{H_{\mathfrak{s}}/Z_{\mathfrak{s}}} & \xrightarrow{\cong} & Laf_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(Z_{\mathfrak{s}}) & \xrightarrow{\cong} & \mathrm{Spec}(Z_{\mathfrak{s}^\vee}) \end{array}$$

where the horizontal arrows are isomorphisms and the vertical arrows are the finite projections of Theorem 1.1.1.

In the first subsection, we recall the definition of enhanced L-parameters and their cuspidality from [AMS18, §6]. In the second subsection, we show that $\Omega_{\mathfrak{s}^\vee} = \mathrm{Spec}(Z_{\mathfrak{s}^\vee})$ is a Bernstein variety on the Galois side ie. it parametrizes the elements of $\mathfrak{s}^\vee$ in the same way as $\Omega_{\mathfrak{s}}$ does for $\mathfrak{s}$ (see Section 2). Finally, we compare our conjecture to other versions of Local Langlands.

1.6.1 Enhanced L-parameters

Let $G^\vee_{\mathrm{ad}} := G^\vee/Z(G^\vee)$ be the adjoint group of $G^\vee$ and $G^\vee_{\mathrm{sc}}$ its simply connected cover. The groups $Z_{G^\vee}(\phi(W'_F))Z(G^\vee)/Z(G^\vee), Z_{G^\vee}(\phi(W_F))Z(G^\vee)/Z(G^\vee)$ can be naturally considered as subgroups of $G^\vee_{\mathrm{ad}}$, and we define $\mathcal{G}'_\phi, \mathcal{G}_\phi \leq G^\vee_{\mathrm{sc}}$ to be their inverse images under the quotient map $G^\vee_{\mathrm{sc}} \to G^\vee_{\mathrm{ad}}$.

After [Art06], [AMS18], we define

$$S_\phi := \pi_0(\mathcal{G}'_\phi).$$

Definition 1.6.5. *We call a representation ρ of S_ϕ an enhancement of ϕ. A pair (ϕ, ρ) where ϕ is an L-parameter and ρ an enhancement of ϕ is called an enhanced L-parameter.*

Not all enhancements are relevant for the bijective Local Langlands correspondence. In particular, given a Langlands parameter ϕ there is a natural group homomorphism $Z(G_{\mathrm{sc}}^\vee)^{W_F} \to Z(S_\phi)$. By Schur's Lemma, ρ acts by a character on the center $Z(S_\phi)$, and therefore we can define a character ζ_ρ of $Z(G_{\mathrm{sc}}^\vee)^{W_F}$.

Let $\gamma \in H^1(F, G_{\mathrm{ad}})$ be the Galois cohomology class parametrizing G as an inner twist of its quasi-split form. We recall an alternate description of $H^1(F, G_{\mathrm{ad}})$ due to Kottwitz.

Proposition 1.6.6. *[Kot84, Proposition 6.4] There is a natural group isomorphism*

$$\kappa : H^1(F, G_{\mathrm{ad}}) \cong \mathrm{Hom}(Z(G_{\mathrm{sc}}^\vee)^{W_F}, \mathbb{C}).$$

Definition 1.6.7. *[AMS18, Definition 6.7] An L-parameter ϕ for G is called G-relevant if for every parabolic subgroup ${}^LP \leq {}^LG$ such that $\phi(W_F) \subseteq {}^LP$, we have that ${}^LP$ comes from a parabolic subgroup $P \leq G$ defined over F.*

An enhanced L-parameter (ϕ, ρ) is called G-relevant if

$$\kappa(\gamma) = \zeta_\rho$$

under the Kottwitz isomorphism.

Remark 1.6.8. *If (ϕ, ρ) is a relevant enhanced L-parameter, then ϕ is a relevant L-parameter by [AMS18, Proposition 6.8].*

The group $G^\vee$ acts on the set of enhanced L-parameters by $g \cdot (\phi, \rho) := (g\phi g^{-1}, {}^g\rho)$. The action of $G^\vee$ preserves relevance.

Definition 1.6.9. *We denote by $\Phi_e({}^LG)$ the set of equivalence classes under $G^\vee$-conjugation* *of enhanced L-parameters for G. We denote by $\Phi_e(G)$ the set of equivalence classes under $G^\vee$-conjugation of relevant enhanced L-parameters for G.*

Conjecture 1.6.10 (Weak Local Langlands Conjecture)**.** *There is a bijection of sets*

$$LLC : \mathrm{Irr}(G) \xrightarrow{\cong} \Phi_e(G).$$

Remark 1.6.11. *Even though the above is a form of bijective Local Langlands, we call it weak because we normally require from a Local Langlands correspondence to satisfy certain natural properties, see for example [Hai14, §5.2], [Bor79].*

To provide a notion of cuspidality for Langlands parameters, a different description of the group S_ϕ is helpful.

Proposition 1.6.12. *We denote by $A_\phi := \pi_0(Z_{\mathcal{G}_\phi}(u_\phi))$ the finite group of connected components of the centralizer. Then, there is an isomorphism*

$$A_\phi \cong S_\phi.$$

We need the following proposition which is a basic tool in the construction of the generalized Springer correspondence.

Proposition 1.6.13. *[AMS18, §2] For any reductive group G, there is a natural bijection between the set of G-conjugacy classes of pairs (u,ρ) and the set of pairs $(\mathcal{C}_u^G, \mathcal{F})$ where $\mathcal{C}_u^G$ is the G-conjugacy class of a unipotent element and $\mathcal{F}$ an irreducible G-equivariant local system in $\mathcal{C}_u^G$. The bijection sends a pair $(\mathcal{C}_u^G, \mathcal{F})$ to any representative u and the representation of $A_G(u)$ on the stalk $\mathcal{F}_u$.*

Definition 1.6.14. *Let $\phi' : W_F \to {}^LG$ be a Langlands parameter. Let ${}^LM$ be a Levi subgroup containing its image $\phi(W_F')$. Then ϕ is called discrete for ${}^LM$ if there is no smaller Levi containing $\phi(W_F')$, and ϕ is relevant for ${}^LM$.*

With these tools in place we can define cuspidality for enhanced L-parameters.

Definition 1.6.15. *A G-equivariant local system is called cuspidal if it does not occur as a constituent in an induction from another local system.*

An enhanced Langlands parameter (ϕ, ρ) for ${}^LM \leq {}^LG$ is called cuspidal if ϕ is discrete and after defining $u_\phi = \phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$, the pair (u_ϕ, ρ) corresponds to a cuspidal local system by applying Proposition 1.6.13 for $\mathcal{G}_\phi$.

Remark 1.6.16. *Lusztig's induction and restriction functors for G-equivariant sheaves provide the needed analog for parabolic induction and restriction of smooth representations. The definition works because the complex dual group is defined over characteristic 0, therefore the decomposition theorem for perverse sheaves is available [BBD82].*

1.6.2 Bernstein theory on the Galois side

Let ${}^LM$ be a Levi subgroup of ${}^LG$ and $X_{\mathrm{nr}}({}^LM) = (Z_{M^\vee \rtimes I_F})^\circ$ where I_F is the inertia subgroup of the Weil group W_F. The natural action of $X_{\mathrm{nr}}({}^LM)$ on $\Phi_e^c(M)$ given by $g \cdot (\phi, \rho) = (g\phi g^{-1}, {}^g\rho)$ where g acts by conjugation on the target of ϕ and on the source of ρ preserves cuspidality. Let $\Phi_e^c(M) \leq \Phi_e(M)$ denote the subset of cuspidal equivalence classes of enhanced L-parameters under $G^\vee$-conjugation. To an enhanced L-parameter $l = (\phi, \rho)$ for G we can attach its cuspidal support $\mathbf{sc}(l) := [(M, l^c)]$ where ${}^LM$ is a minimal Levi where l^c is defined and $l^c \in \Phi_e^c(M)$ such that the G-equivariant local system $\mathcal{F}_l$ corresponding to l under Proposition 1.6.13 occurs as a constituent in the induction of $\mathcal{F}_{l^c}$. The cuspidal support is well-defined up to $G^\vee$-conjugation [AMS18, §7].

We denote by $\mathfrak{s}^\vee$ the orbit of $[({}^LM, l^c)]$ under the action of $X_{\mathrm{nr}}({}^LM)$ on the second term, and by $\mathfrak{B}^\vee(G)$ the set of all possible orbits. We denote by $\Phi_e^{\mathfrak{s}^\vee}(G)$ the preimage $\mathbf{sc}^{-1}(\mathfrak{s}^\vee)$

of $\mathfrak{s}^\vee$ under the cuspidal support map, which provides us with the partition

$$\Phi_e(G) \cong \bigsqcup_{\mathfrak{s}^\vee \in \mathfrak{B}^\vee(G)} \Phi_e^{\mathfrak{s}^\vee}(G)$$

of [AMS18].

Let $l = (\phi, \rho)$ be a representative for the class $\mathfrak{s}^\vee$. We denote by $X_{\mathrm{nr}}(M^\vee, l) \subseteq X_{\mathrm{nr}}(M^\vee)$ the finite subgroup stabilizing l up to $M^\vee$-conjugacy. Then $T_{\mathfrak{s}^\vee} := X_{\mathrm{nr}}(M^\vee)/X_{\mathrm{nr}}(M^\vee, l) \cong \Phi_e^c(M)$ is an algebraic torus.

Similarly to the proof of Proposition 1.2.16, $T_{\mathfrak{s}^\vee}$ parametrizes cuspidal pairs up to $M^\vee$-conjugacy. We denote by $W_{\mathfrak{s}^\vee} \subseteq W(M^\vee)$ the subgroup of the finite Weyl group stabilizing l up to $G^\vee$-conjugacy.

We recall the main theorem of [AMS21] in the notation of Section 2.

Theorem 1.6.17. *There are parameters λ, λ^* and a specialization $H_{\mathfrak{s}^\vee}$ of the twisted affine Hecke algebra $\mathcal{H}(T_{\mathfrak{s}^\vee}, W_{\mathfrak{s}^\vee}, \lambda, \lambda^*, \vec{\mathbf{z}})$ such that*

$$\Phi_e^{\mathfrak{s}^\vee} \cong \mathrm{Irr}(H_{\mathfrak{s}^\vee})$$

We can now define our analog of the Bernstein variety.

Lemma 1.6.18 (Bernstein variety on the Galois side)**.** *The algebraic variety $\Omega_{\mathfrak{s}^\vee}(G) = T_{\mathfrak{s}^\vee}/W_{\mathfrak{s}^\vee}$ parametrizes the equivalence classes of cuspidal pairs in $\mathfrak{s}^\vee$.*

The ring of regular functions

$$Z_{\mathfrak{s}^\vee} = \mathcal{O}(\Omega_{\mathfrak{s}^\vee}) = \mathcal{O}(T_{\mathfrak{s}^\vee})^{W_{\mathfrak{s}^\vee}}$$

of the Bernstein variety is isomorphic to $Z(H_{\mathfrak{s}^\vee})$.

The finite projection $p : \mathrm{Laf}_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}}(\mathbb{C}) \to \Omega_{\mathfrak{s}^\vee}(\mathbb{C})$ of Theorem 1.1.1 restricted to $\mathrm{iLaf}_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}}$ is the cuspidal support map.

Proof. For the first part, by the discussion of the previous paragraph, equivalence classes in $\mathfrak{s}^\vee$ are parametrized by $T_{\mathfrak{s}^\vee}$ up to the transitive action of $W_{\mathfrak{s}^\vee}$. As the quotient of a torus by a finite group, $\Omega_{\mathfrak{s}^\vee}$ admits a natural structure of an algebraic variety.

The second part follows from the explicit structure of $H_{\mathfrak{s}^\vee}$ in Theorem 1.6.17 and Lemma 1.2.48.

If $l = (\phi, \rho)$ is an enhanced L-parameter identified with a closed point of $\mathrm{iLaf}_{H_{\mathfrak{s}^\vee}/Z_{\mathfrak{s}^\vee}}$, then $p(l) \in \Omega_{\mathfrak{s}^\vee}$ is a cuspidal support such that l is a constituent of the induction of $p(l)$. Therefore, $p(l) = \mathbf{sc}(l)$ by uniqueness of cuspidal support. □

1.6.3 Relations between the conjectures

We relate some known conjectures with Conjecture 1.6.4.

Proposition 1.6.19. *Conjecture 1.6.4 implies Conjecture 1.6.10.*

Proof. By Proposition 1.4.3, Conjecture 1.6.4 provides us with a canonical isomorphism

$$\mathrm{iLaf}_{H_{\mathfrak{s}}} \cong \mathrm{iLaf}_{H_{\mathfrak{s}^\vee}},$$

which at the level of $\mathbb{C}$-points provides us with a canonical bijection. □

Remark 1.6.20. *Notice that by Lemma 1.6.18, the compatibility with central characters required in [Bor79] for a Local Langlands Correspondence also follows from Conjecture 1.6.4.*

Conjecture 1.6.21. *For any reductive group G, it is true that*

$$H_{\mathfrak{s}} \overset{a}{\sim} H_{\mathfrak{s}^\vee}.$$

Proposition 1.6.22. *Conjecture 1.6.21 implies Conjecture 1.6.4. In particular, Conjecture 1.6.4 is true for quasi-split classical groups and their pure inner forms, ie. symplectic groups, (special) orthogonal groups, general (s)pin groups and unitary groups.*

Proof. By Proposition 1.4.3, the first part follows. The fact that Conjecture 1.6.21 is true for pure inner forms of quasi-split classical groups is the content of [AMS22]. □

CHAPTER 2

CHARACTER SHEAVES

2.1 Introduction

2.1.1 Summary

Character sheaves on a reductive group G are certain irreducible G-equivariant perverse sheaves introduced by Lusztig as a geometrization of characters of a representation in the series of papers [Lus85a], [Lus85b], [Lus85c], [Lus86a], [Lus86b]. Roughly speaking, in characteristic 0, we start with a local system $\mathcal{L}$ with finite monodromy on a torus T of a Borel subgroup B that is invariant under some element $w \in W$, push it forward to a B-equivariant local system on BwB, and then Lusztig's induction functor gives a G-equivariant perverse sheaf $K_w^{\mathcal{L}}$ on G. Character sheaves are the irreducible constituents of all possible $K_w^{\mathcal{L}}$. In characteristic p, a condition is added to this definition, more precisely that the local system $\mathcal{L}$ we start from is a Kummer local system. For the precise definitions, see [MS89, §2] or Section 3.

In the finite field case, taking the trace of Frobenius on character sheaves retrieves the irreducible characters, and the category generated by character sheaves behaves similarly to the category of representations in infinite fields, where the theory of characters is not well-defined.

Let $\mathcal{N}$ be the subalgebra of nilpotent elements of $\mathfrak{g}$. It can be embedded in $\mathfrak{g}^*$ using the Killing form, and then we define $G \times \mathcal{N} \subseteq T^*G \cong G \times \mathfrak{g}^*$ to be the *nilpotent cone*. Independently, Ginzburg and Mirkovic and Vilonen showed that, for k an algebraically closed field of characteristic zero, character sheaves are exactly the G-equivariant irreducible perverse sheaves whose singular support is a subvariety of the nilpotent cone [Gin89], [MV88]. Ginzburg employed the Riemann-Hilbert correspondence and the classical notion of singular support for D-modules, whereas Mirkovic and Vilonen used the microlocal definition of singular support of constructible sheaves on varieties over characteristic 0 defined in [KS90], which circumvents the use of D-modules.

Beilinson generalized the notion of singular support for constructible sheaves defined over any characteristic [Bei16]. While singular support in characteristic p enjoys similar functorial properties to the characteristic 0 case, certain aspects needed in the proof of Mirkovic and Vilonen, like conormality of the singular support to its base, no longer work in characteristic p, [Bei16, Example 1.6].

In this paper, we define a category of *tame* perverse sheaves and study their functorial properties. This notion captures most sheaves used in the construction of character sheaves, and tame perverse sheaves behave similarly enough to the characteristic 0 case. We then adapt Mirkovic and Vilonen's proof to show the following, without any assumption on $\mathrm{char}(k)$.

Theorem 2.1.1. *Let G be a reductive group over an algebraically closed field k. Then the singular support of a character sheaf is a subvariety of $G \times \mathcal{N}$.*

2.1.2 Outline

We define a general notion of tame perverse sheaves, and show that the category they form is particularly well-behaved, using Kerz and Schmidt's results about tameness of étale coverings [KS10].

Upon trying to adapt the proof of Mirkovic and Vilonen to positive characteristic, we stumble in the following problems.

First of all, Mirkovic and Vilonen use in an essential way the conormality of singular support to its base in characteristic 0, which does not remain true in characteristic p. We show that conormality of the singular support is true for tame perverse sheaves, upon restrictions on the ramification divisor.

Second, we need to show that certain operations of sheaves involved in the construction of character sheaves preserve tameness. We show preservance of tameness under various operations, and in particular we will define a class of morphisms called *tamely smooth*, with the property that the pullback under them is tame if and only if our initial perverse sheaf was tame.

The paper is organized as follows: In Section 2, we define and study the category of tame perverse sheaves. In Section 3, we prove Theorem 2.1.1.

2.2 Tame perverse sheaves

2.2.1 Definition of a tame perverse sheaf

Let X be a smooth variety over an algebraically closed field k. All sheaves we consider are assumed to be either l-adic for $l \neq \mathrm{char}(k)$, or complex.

E will denote any constant sheaf. We recall a well-known definition.

Definition 2.2.1. *$\mathcal{L}$ is a local system on U with finite monodromy if there exists a finite étale covering map $f : \tilde{U} \to U$ such that $f^*\mathcal{L} = E$.*

We recall the following definition.

Definition 2.2.2. *Let $\bar{C}$ be a proper, connected and regular curve of finite type over* $\mathrm{Spec}(k)$ *and $C \subseteq \bar{C}$ an open subscheme. Every point $x \in \bar{C} \smallsetminus C$ defines a valuation on $k(C)$. An étale covering map of curves $C' \to C$ is called* tame *if for every $x \in \bar{C} \smallsetminus C$ the valuation v_x is tamely ramified in $k(C') \mid k(C)$.*

We also recall one of several equivalent definitions for tameness of an étale covering in [KS10, §4].

Definition 2.2.3. *An étale covering $Y \to X$ is called* tame *if for every morphism $C \to X$ the base change $Y \times_X C \to C$ is tame.*

The following definitions are motivated by Definition 2.2.3.

Definition 2.2.4. *A local system $\mathcal{L}$ with finite monodromy will be called* tame *if the étale covering $f : \tilde{U} \to U$ in Definition 2.2.1 can be taken to be tame. Equivalently, if the étale covering $f : \tilde{U} \to U$ representing $\mathcal{L}$ is tame.*

Definition 2.2.5. *An irreducible perverse sheaf $IC(U, \mathcal{L})$ on a smooth variety X is called* tame *if $\mathcal{L}$ is a tame local system on U. A perverse sheaf on X is called* tame *if all its irreducible constituents are tame.*

2.2.2 Properties

In this subsection we show properties of tameness that will be used later.

Lemma 2.2.6. *The category $Perv_{tame}(X) \subseteq Perv(X)$ consisting of the tame perverse sheaves and the morphisms between them is an abelian subcategory of the category of perverse sheaves.*

Proof. Trivial by checking the irreducible constituents. □

Lemma 2.2.7. *Tameness of an étale covering is stable under arbitrary base change.*

Proof. Let $f : Y \to X$ be a tame étale covering and $g : Z \to X$ a morphism. Then for an arbitrary morphism $C \to Z$ where C is a regular curve, we get the following diagram

$$\begin{tikzcd}
Y \times_X Z \times_Z C \arrow[r] \arrow[d] & C \arrow[d] \\
Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}$$

It is enough to prove that the upper morphism of curves is tame. This follows by $Y \times_X Z \times_Z C \cong Y \times_X C$ and the tameness of $Y \to X$. □

As a consequence, smooth pullbacks preserve tameness.

Lemma 2.2.8. *Let $f : X \to Y$ be a smooth morphism, and $\mathcal{F}$ a tame perverse sheaf on Y. Then $f^*\mathcal{F}$ is tame.*

Proof. It is enough to prove the statement for an irreducible perverse sheaf $IC(Y_0, \mathcal{L})$ on Y. Restrict Y to $\bar{Y_0}$ and X to the preimage, and f will remain smooth in the restriction because smoothness is preserved by arbitrary base change. Then the pullback will be $IC(f^{-1}(Y_0), f^*\mathcal{L})$. But for $\mathcal{L}$ there must be a tame étale covering $g : U \to Y$ such that $g^*\mathcal{F} = E$. Therefore, by the diagram

$$\begin{tikzcd} U \times_Y X \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ U \arrow[r, "g"] & Y \end{tikzcd}$$

since by smooth base change $g'^* f^* \mathcal{L} = f'^* g^* \mathcal{L} = f'^* E = E$, g' is an étale covering trivializing $f^*\mathcal{F}$. By Lemma 2.2.7, g' is also tame. □

Tameness is local with respect to the tame site of [HS21].

Lemma 2.2.9. *Let $\mathcal{F}$ be a perverse sheaf on a smooth variety X and $\{U_i\}$ be an étale covering of X such that all maps $p_i : U_i \to X$ are tame. Then $\mathcal{F}$ is tame if and only if all $p_i^*\mathcal{F}$ are tame sheaves. In particular, if $p : Y \to X$ is an étale map that is a tame covering of its image, and $p^*\mathcal{F}$ is tame, then $\mathcal{F}$ is tame.*

Proof. The only if direction follows directly Lemma 2.2.7. For the if direction, we use descent and the Lemma 2.2.7 to get a tame étale covering $p : U \to X$ such that $p^*\mathcal{F}$ is tame. Then if $g : V \to U$ is a tame covering that trivializes $p^*\mathcal{F}$, $p \circ g$ is a tame étale covering trivializing $\mathcal{F}$.

For the second assertion, we claim that it is enough to check tameness of a sheaf after its restriction to a Zariski open subset, and then we apply the first assertion.

Indeed, after we base change to any curve, it is enough to notice that tameness on a curve can be checked on any open subset, which is well-known. □

In the case of a smooth projection, we have a converse.

Lemma 2.2.10. *Let Y smooth and $p : X \times Y \to X$ be the first projection, and $\mathcal{F}$ a perverse sheaf on X. Then $p^*\mathcal{F}$ is tame if and only if $\mathcal{F}$ is.*

Proof. The if direction is immediate by Lemma 2.2.8. For the only if direction, we can assume $\mathcal{F} = IC(U, \mathcal{L})$ and then we have that $p^*\mathcal{F} = IC(X \times U, \mathcal{O}_X \otimes \mathcal{L})$ is tame. A minimal trivialization of $p^*\mathcal{F}$ in particular trivializes $\mathcal{L}$ and thus $\mathcal{F}$ is also tame. □

On a torus, tame local systems coincide with Kummer local systems. Indeed, for $n \in \mathbb{N}$, let $n : T \to T$ be the n-th power isogeny $n(t) = t^n$.

Lemma 2.2.11. *For a torus T, a local system $\mathcal{L}$ is tame if and only if it $n^*\mathcal{L} \cong E$ for some n coprime to p.*

Proof. Let $R = k[x_1^{\pm}, \dots, x_k^{\pm}]$ and $T = \mathrm{Spec}R$. Since $n^*\mathcal{L} = E$, $\mathcal{L}$ trivializes under the covering the covering $n : \mathrm{Spec}R[t_1, \dots, t_k]/(t_1^n - x_1, \dots, t_k^n - x_k) \to \mathrm{Spec}R$, which is a standard tame covering since $(n, p) = 1$. If it is tame, then it must trivialize under some tame covering, which by the relative Abhyankar Lemma [Gro71, Expose XIII, Proposition 5.1] we can take to be of the form $f : \mathrm{Spec}R[t_1, \dots, t_k]/(t_1^{n_1} - x_1, \dots, t_k^{n_k} - x_k)$, where $(n_i, p) = 1$ for all $i = 1, \dots, k$. Defining $n = \prod n_i$, we see that we can write n as a composition $g \circ f$, and therefore n also trivializes $\mathcal{L}$, since $n^*\mathcal{L} = g^* \circ f^*(\mathcal{L}) = g^*E = E$. □

We now show that conormality of the singular support still holds in characteristic p for tame perverse sheaves upon conditions on the geometry of the ramification divisor.

Definition 2.2.12. *Let X be a variety and $D \subset X$ a closed subvariety. A map $f : \tilde{X} \to X$ will be a called a uniform resolution of D if*

1. *$\tilde{X}$ is smooth and f is proper and birational.*

2. *$f^{-1}(D)$ is a simple normal crossings divisor in $\tilde{X}$.*

3. f is a stratified submersion on tangent spaces for the stratification $D = \bigsqcup_{i\in I} D_i$ by smooth strata of codimension i.

Lemma 2.2.13. *Let X be a smooth variety over an algebraically closed field k. If $\mathcal{F}$ is a tame perverse sheaf on X such that for every irreducible constituent $IC(U,\mathcal{L})$ we have that $D := \bar{U} \setminus U$ has a uniform resolution, then $SS(\mathcal{F})$ is conormal to its base.*

Proof. Assume without loss of generality that $\mathcal{F} := IC(U,\mathcal{L})$ is irreducible. Shrink U by intersecting with the open locus where f is an isomorphism. Consider the diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "\tilde{j}"] \arrow[d, "g"] & \tilde{X} \arrow[d, "f"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

where $j, \tilde{j}$ are inclusions and g is the restriction of f to $f^{-1}(U)$.

If D is already a simple normal crossings divisor, since $SS(\mathcal{F}) \subseteq SS(j_!\mathcal{L})$ by [Bei16, Theorem 1.4(ii)], the assertion follows from a combination of [Sai16, Lemma 3.3] and [KS10, Theorem 4.4].

For the general case, we define $\mathcal{G} := \tilde{j}_! g^*\mathcal{L}$, and notice that

$$j_!\mathcal{L} \cong f_* f^* j_!\mathcal{L} \cong f_*\mathcal{G}$$

where the first isomorphism is true because f is an isomorphism over U, and the second by base change. Therefore, by [Bei16, Lemma 2.2]

$$SS(\mathcal{F}) \subseteq SS(j_!\mathcal{L}) = SS(f_*\mathcal{G}) \subseteq f_\circ SS(\mathcal{G}),$$

and $SS(\mathcal{G})$ is conormal since $f^{-1}(D)$ is a simple normal crossings divisor and $g^*\mathcal{L}$ is tame.

Assume $SS(\mathcal{F})$ was not conormal. Then there exists $z \in D_i$ and w not conormal to D_i, such that $w \in SS(\mathcal{F})$. Therefore, there exists v tangent to D_i such that $\langle w, v\rangle \neq 0$. By assumption, there exists v' tangent to $f^{-1}(D_i)$ such that $\langle df(w), v'\rangle \neq 0$, and by conormality of $SS(\mathcal{G})$ we have a contradiction. $\square$

Remark 2.2.14. *For the case of character sheaves treated in this paper we need conormality for the Bruhat stratification, which admits the Bott-Samelson resolution. The Bott-Samelson resolution is uniform by equivariance.*

Notice that, in general, pushforwards do not preserve tameness even under very strong assumptions.

Example 2.2.15. *Consider a projective wildly ramified covering of curves $f : X \to Y$. This decomposes as $X \xrightarrow{i} Y \times \mathbb{P}^1 \xrightarrow{p} Y$. $\mathcal{F} = i_* E$ is a tame sheaf, while $p_* \mathcal{F} = f_* E$ is not. Notice that p is a smooth proper map.*

2.2.3 Tamely smooth morphisms

We define a refinement of smoothness that is useful for studying tameness. First, we recall an equivalent definition for smoothness [Sta22, Tag 054L].

Lemma 2.2.16 (Smooth maps are étale locally trivial bundles)**.** *A map $f : X \to Y$ is smooth if an only if for every point $x \in X$ there exist Zariski open neighborhoods U, V around $x, f(x)$ such that f decomposes as*

$$\begin{tikzcd} X \arrow[d, "f"] & U \arrow[l] \arrow[r, "\pi"] \arrow[d, "f'"] & \mathbb{A}^n \times V \arrow[dl, "p"] \\ Y & V \arrow[l] & \end{tikzcd}$$

where π is an étale map.

Motivated by Lemma 2.2.16, we define the notion of a *tamely smooth morphism.* Essentially, tamely smooth morphisms are tamely locally trivial bundles for the tame site defined in [HS21].

Definition 2.2.17. *A smooth morphism $f : X \to Y$ is called* tamely smooth *if the étale maps π in Lemma 2.2.16 can be taken to be tame étale coverings of their image.*

The following properties are easy, but we include the proof for completeness.

Lemma 2.2.18. *We have the following properties.*

(a) Open immersions are tamely smooth.

(b) Tame smoothness is preserved by base change.

(c) The composition of two tamely smooth morphisms is tamely smooth.

Proof.

(a) If $j : U \to X$ is an open immersion, we can just take for every $u \in U$ the neighborhoods $U \subseteq U$ and $U \subseteq X$, and j becomes an isomorphism.

(b) Follows by taking base change of the diagram in Lemma 2.2.16.

(c) Let $f : X \to Y, g : Y \to Z$ be tamely smooth, and consider the following diagram, where we have taken refinements of the neighborhoods in Lemma 2.2.16 and the upper right diagram is base change.

$$\begin{tikzcd} X \arrow[d] & U \arrow[l, hook'] \arrow[r] \arrow[d] & V \times \mathbb{A}^n \arrow[r] \arrow[dl] & W \times \mathbb{A}^{m+n} \arrow[dl] \\ Y \arrow[d] & V \arrow[l, hook'] \arrow[r] \arrow[d] & W \times \mathbb{A}^m \arrow[dl] & \\ Z & W \arrow[l, hook'] & & \end{tikzcd}$$

Then the fact that $V \to W \times \mathbb{A}^m$ is tame implies that its base change is tame, and then by the fact that the composition of tame étale coverings is tame we can conclude.

□

The next Lemma is the motivation behind the introduction of tamely smooth morphisms.

Lemma 2.2.19. *Let $f : X \to Y$ be a tamely smooth morphism and $\mathcal{F}$ a perverse sheaf on Y such that $f^*\mathcal{F}$ is tame. Then $\mathcal{F}$ is tame.*

Proof. Using the notation from the diagram in Lemma 2.2.16, it is enough to show the statement for f' due to Lemma 2.2.9 . But then if $f'\mathcal{F} = \pi^* p^* \mathcal{F}$ is tame, $p^*\mathcal{F}$ is tame by Lemma 2.2.9 , and therefore $\mathcal{F}$ is tame by Lemma 2.2.10 . $\square$

2.3 Proof of the main theorem

We adapt Mirkovic and Vilonen's proof, using the results of the previous Section where needed, to prove Theorem 2.1.1. First of all, we recall the definition of character sheaves. In particular, we recall Lusztig's induction functor, a sheaf analogue of the induction functor for representations, or characters. Indeed, in the finite field case, taking the trace of Frobenius to reduce to classical characters, commutes with induction.

Let A be a connected algebraic group acting on a variety X. For any connected subgroup B, consider the following commutative diagram.

$$\begin{tikzcd} A \times X \arrow[r, "\nu"] \arrow[d, "a"] & A/B \times X \arrow[d, "p"] \\ X & X \end{tikzcd}$$

given by

$$\begin{tikzcd} (a,x) \arrow[r, "\nu"] \arrow[d, "a"] & (aB,x) \arrow[d, "p"] \\ a^{-1}\cdot x & x. \end{tikzcd}$$

For $\mathcal{F} \in D_B(X)$, there exists a unique $\tilde{\mathcal{F}} \in D_A(A/B \times X)$ such that $a^*\mathcal{F} = \nu^*\tilde{\mathcal{F}}$.

Definition 2.3.1. *The functor* $\Gamma_B^A \mathcal{F} = p_* \tilde{\mathcal{F}}$ *is called the* induction functor*.*

We now recall the definition of character sheaves following [MS89]. Let G be a reductive group and fix a Borel B. Let T be a maximal torus and $B = TN$ for the unipotent radical. Let W be the Weyl group of B, T and $n = dim G/B$. Let $w \in W$. We consider $G_w = BwB$ and $Y_w = \overline{G_w}$. Choosing a representative $\dot{w} \in W$, there is an isomorphism $U_w \times T \times U \cong G_w$ given by $(u,t,u') \to u\dot{w}tu'$, so we have a natural projection $pr : G_w \to T$ defined by $pr(u,t,u') = t$.

For any Kummer local system $\mathcal{L}_\xi$ on T, we define $\mathcal{L}_{\xi,\dot{w}} = pr^*\mathcal{L}_\xi$, and set $A_{\xi,\dot{w}} = IC(G_w, \mathcal{L}_{\xi,\dot{w}})$. $A_{\xi,\dot{w}}$ up to isomorphism does not depend on the choice of $\dot{w}$, so we will write $A_{\xi,w}$.

Definition 2.3.2. *The irreducible constituents of the complexes* $K_w^{\mathcal{L}} = \Gamma_B^G[A_{\xi,w}]$ *where* $w \in W$ *and* $\mathcal{L}_\xi$ *is a Kummer local system on* T *fixed by the* w*-action, are called* character sheaves.

By Lemmas 2.2.11 and 2.2.8, the sheaves $A_{\xi,w}$ are tame.

To prove Theorem 2.1.1, we need the equivalent of [MV88, Lemma 1.2] for characteristic p, and then to be able to follow partially the proof of [MV88, Theorem 2.7]. For the latter part, we will use Lemma 2.2.13 and the Bott-Samelson resolution.

Lemma 2.3.3. *For* $\mathcal{F} \in D_B(X)$*, we have*

$$SS(\Gamma_B^G \mathcal{F}) \subseteq \overline{G \cdot SS(\mathcal{F})}.$$

Proof. Let pr_2 be the second projection from either $T^*(G \times X), T^*(G/B \times X)$ to T^*X.

We have $pr_2(SS(\tilde{\mathcal{F}})) = pr_2(SS(\nu^*\tilde{\mathcal{F}})) = pr_2(SS(a^*\mathcal{F}))$.

The singular support of a pullback via a smooth morphism behaves as in characteristic 0 by [Bei16, Lemma 2.2] or [Sai16] so we similarly have $pr_2(SS(\tilde{\mathcal{F}})) = G \cdot \mathcal{F}$.

Now, we need to prove $SS(\Gamma_B^G \mathcal{F}) = SS(p_*\tilde{\mathcal{F}}) \subseteq \overline{pr_2(SS(\tilde{\mathcal{F}}))}$.

But G/B is a flag variety and therefore proper, so p is proper being a base change of a proper morphism. Therefore, we conclude by [Bei16, Lemma 2.2]. □

We are now ready to prove Theorem 2.1.1.

Proof. First, we show that $A_{\xi,w}$ has nilpotent singular support.

Under the identification $T^*G \cong G \times \mathfrak{g}$, the fiber at e of the conormal bundle at the Bruhat cell B is n. Therefore, the fiber at g of the Bruhat cell BgB is $n \cap {}^g n$ where the upper script denotes the adjoint action. So it is enough to prove that $A_{\xi,w}$ has conormal singular support. But $A_{\xi,w}$ is a tame perverse sheaf constructible with respect to the Bruhat stratification, so its irreducible constituents are supported on Bruhat cells $Y_w = \overline{BwB}$. Choose a reduced expression $w = s_1 \dots s_n$. We get conormality by Lemma 2.2.13, using the Bott-Samelson resolution

$$Y_{s_1} \times_B \cdots \times_B Y_{s_n} \to Y_w$$

(see, for example, [Lus85a]).

Finally, $G \times \mathcal{N}$ is closed and invariant for the G-actions of translation and conjugation, thus by Lemma 2.3.3 Γ_B^G preserves nilpotency of singular support. $\square$

REFERENCES

[ABD+66] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Alexander Grothendieck, Pierre Gabriel, Michel Raynaud, and Jean-Pierre Serre. *Schémas en groupes.* Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques. Institut des Hautes Études Scientifiques, Paris, 1963/1966.

[AMS18] Anne-Marie Aubert, Ahmed Moussaoui, and Maarten Solleveld. Generalizations of the Springer correspondence and cuspidal Langlands parameters. *Manuscripta Mathematica*, 157(1-2):121–192, 2018.

[AMS21] Anne-Marie Aubert, Ahmed Moussaoui, and Maarten Solleveld. Affine Hecke algebras for Langlands parameters. arXiv:1701.03593, 2021.

[AMS22] Anne-Marie Aubert, Ahmed Moussaoui, and Maarten Solleveld. Affine Hecke algebras for classical p-adic groups. arXiv:1701.03593, 2022.

[Art06] James Arthur. A Note on L-packets. *Pure and Applied Mathematics Quarterly*, 2:199–217, 2006.

[BBD82] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. *Astérisque*, 100(1), 1982.

[BBK18] Joseph Bernstein, Roman Bezrukavnikov, and David Kazhdan. Deligne-Lusztig duality and wonderful compactification. *Selecta Mathematica*, 24(1):7–20, 2018.

[BD84] Joseph N Bernstein and Pierre Deligne. Le "centre" de Bernstein. In *Representations of reductive groups over a local field*, pages 1–32, 1984.

[Bei16] Alexander Beilinson. Constructible sheaves are holonomic. *Selecta Mathematica*, 22(4):1797–1819, 2016.

[Ber92] J. Bernstein. Representations of p-adic groups, 1992.

[BK98] CJ Bushnell and PC Kutzko. Smooth representations of reductive p-adic groups: structure theory via types. *Proceedings of the London Mathematical Society*, 77(3):582–634, 1998.

[Bor79] Borel. *Automorphic Forms, Representations and L-Functions.* Automorphic Forms, Representations and L-functions: Proceedings of the Symposium in Pure Mathematics of the American Mathematical Society Held at Oregon State University, Corvallis, Oregon, July 11-August 5, 1977. American Mathematical Society, 1979.

[Bou89] N. Bourbaki. *Algebra: Chapters 1-3.* Algebra I. Springer-Verlag, 1989.

[BZ76] Joseph Bernstein and Andrei Vladlenovich Zelevinskii. Representations of the group GL(n,F) where F is a non-Archimedean local field. *Uspekhi Matematicheskikh Nauk*, 31(3):5–70, 1976.

[Cas86] J. W. S. Cassels. *Local Fields*. London Mathematical Society Student Texts. Cambridge University Press, 1986.

[EE95] D. Eisenbud and P.D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995.

[Gin89] Victor Ginzburg. Admissible modules on a symmetric space. *Astérisque*, 173(174):199–255, 1989.

[Gro61] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert. In *Séminaire Bourbaki : années 1960/61, exposés 205-222*, number 6 in Séminaire Bourbaki. Société mathématique de France, 1961. talk:221.

[Gro71] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[Hai14] Thomas Haines. The stable Bernstein center and test functions for Shimura varieties. In *Automorphic Forms and Galois Representations*, volume 415 of *London Mathematical Society Lecture Note Series*, pages 118–186. Cambrige University Press, 2014.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HKP09] Thomas J. Haines, Robert E. Kottwitz, and Amritanshu Prasad. Iwahori-hecke algebras. arXiv:math/0309168, 2009.

[HS21] Katharina Hübner and Alexander Schmidt. The tame site of a scheme. *Inventiones mathematicae*, 223:379–443, 2021.

[Hym68] Bass Hyman. *Algebraic K-theory*. WA Benjamin Incorporated, 1968.

[IM65] N. Iwahori and H. Matsumoto. *On Some Bruhat Decomposition and the Structure of the Hecke Rings of P-adic Chevalley Groups*. Publications mathématiques. Institut des hautes études scientifiques, 1965.

[Jac45] Nathan Jacobson. Structure theory of simple rings without finiteness assumptions. *Transactions of the American Mathematical Society*, 57(2):228–245, 1945.

[Kat81] Shin-ichi Kato. Irreducibility of principal series representations for Hecke algebras of affine type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math*, 28(3):929–943, 1981.

[KL87] D. Kazhdan and G. Lusztig. Proof of the Deligne-Langlands conjecture for Hecke algebras. *Inventiones mathematicae*, 87:153–216, 1987.

[Kot84] Robert E Kottwitz. Stable trace formula: Cuspidal tempered terms. *Duke Mathematical Journal (C)*, 1984.

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on Manifolds*. Springer Berlin Heidelberg, 1990.

[KS10] Moritz Kerz and Alexander Schmidt. On different notions of tameness in arithmetic geometry. *Mathematische Annalen*, 346:641–668, 2010.

[Laf16] Laurent Lafforgue. Le principe de fonctorialité de Langlands comme un problème de généralisation de la loi d'addition, 2016.

[Lam91] T Y Lam. *A first course in noncommutative rings*, volume 131 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, New York, NY, third edition, 2002.

[Lin82] Hartmut Lindel. On the Bass-Quillen Conjecture Concerning Projective Modules Over Polynomial Rings. *Inventiones mathematicae*, 65:319–324, 1981/82.

[Lus85a] George Lusztig. Character sheaves I. *Advances in Mathematics*, 56(3):193–237, 1985.

[Lus85b] George Lusztig. Character sheaves II. *Advances in Mathematics*, 57(3):226–265, 1985.

[Lus85c] George Lusztig. Character sheaves III. *Advances in Mathematics*, 57(3):266–315, 1985.

[Lus86a] George Lusztig. Character sheaves, IV. *Advances in Mathematics*, 59(1):1–63, 1986.

[Lus86b] George Lusztig. Character sheaves, V. *Advances in Mathematics*, 61(2):103–155, 1986.

[MS89] J. G. M. Mars and T. A. Springer. Character sheaves. In *Orbites unipotentes et représentations III. Orbites et faisceaux pervers*, number 173-174 in Astérisque. Société mathématique de France, 1989.

[Mul79] I. Muller. Integrales d'entrelacement pour un groupe de chevalley sur un corps p-adique. In Pierre Eymard, Reiji Takahashi, Jacques Faraut, and Gérard Schiffmann, editors, *Analyse Harmonique sur les Groupes de Lie II*, pages 367–403, Berlin, Heidelberg, 1979. Springer Berlin Heidelberg.

[MV88] Mirkovic and Vilonen. Characteristic varieties of character sheaves. *Inventiones mathematicae*, 93(2):405–418, 1988.

[Nag62] M. Nagata. *Local Rings*. Interscience tracts in pure and applied mathematics. Interscience Publishers, 1962.

[Rob63] Norbert Roby. Lois polynomes et lois formelles en théorie des modules. *Annales Scientifiques de l'Ecole Normale Supérieure. Quatrième Série*, 80(3):213–348, 1963.

[Roc02] Alan Roche. Parabolic induction and the bernstein decomposition. *Compositio Mathematica*, 134(2):113–133, 2002.

[Sai16] Takeshi Saito. The characteristic cycle and the singular support of a constructible sheaf. *Inventiones mathematicae*, 207(2):597–695, Jul 2016.

[Sol21] Maarten Solleveld. Affine Hecke algebras and their representations. *Indagationes Mathematicae*, 32(5):1005–1082, sep 2021.

[Sol22] Maarten Solleveld. Endomorphism algebras and Hecke algebras for reductive p-adic groups. *Journal of Algebra*, 606:371–470, 2022.

[Spr] TA Springer. *Linear algebraic groups*, volume 9. Springer.

[Sta22] The Stacks project authors. The Stacks Project. `https://stacks.math.columbia.edu`, 2022.

[Ste74] Robert Steinberg. *Reductive and semisimple algebraic groups, regular and subregular elements*, pages 76–155. Springer Berlin Heidelberg, Berlin, Heidelberg, 1974.