Geometry of p-adic representations

The Lafforgue variety and generalized discriminants

Kostas I. Psaromiligkos[1,2]

[1] University of Chicago, kostaspsa@math.uchicago.edu
[2] Onassis Foundation

October 2, 2022

Table of Contents

Introduction

Background and Motivation

Let $G(F)$ be a split reductive group over a non-archimedean local field F with ring of integers $\mathcal{O}$ and uniformizer π. Let $q = |\mathcal{O}/\pi\mathcal{O}|$ the cardinality of the residue field. Assume B is a Borel, T a split maximal torus and $W = N(T)/T$ the Weyl group. Think $GL_2(F), GL_n(F), SL_n(F)$, etc. and for GL_n, B the upper triangular matrices, T the diagonal matrices, and $W = S_n$.

We want to study *smooth irreducible* representations of $G(F)$. These are all admissible by a theorem of Bernstein, and there is a finite-to-one parametrization by the *Bernstein variety*.

We will elevate this to an one-to-one parametrization suggested in [1], by an open subscheme of an affine scheme, which we call *Lafforgue variety*.

The Hecke algebra

Let I be an Iwahori subgroup ie. for GL_n,

$$I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \dots & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O}^\times & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ \pi\mathcal{O} & \pi\mathcal{O} & \dots & \mathcal{O}^\times \end{pmatrix}$$

We consider the Hecke algebra H of I-biinvariant locally constant compactly supported distributions on G under convolution. Smooth irreducible representations of G containing an I-fixed vector are equivalent to simple H-modules.

The center $Z(H)$ of H is Cohen-Macaulay, and H is a Cohen-Macaulay module over $Z(H)$, see [2]. Therefore, there is a regular subalgebra A of $Z(H)$ such that H is a finite locally free module over A.

The Hecke algebra

Theorem (Bernstein presentation)

Let $R = \mathbb{C}[X_*(T)]$ *the group algebra of the cocharacter lattice. Then* H *is generated over* R *by elements* $T_w = T_{s_1} \cdots T_{s_n}$ *such that*

$$T_{s_a}^2 = (q-1)T_{s_a} + q \tag{1}$$

$$T_{s_a}\pi = s_a(\pi)T_{s_a} + (q-1)\frac{\pi - s_a(\pi)}{1 - \pi^{-a^\vee}} \tag{2}$$

Example

Let H be the Iwahori-Hecke algebra of $GL_2(\mathbb{Q}_p)$. Then H is generated over $\mathbb{C}[x^{\pm}, y^{\pm}]$ by $1, T_s$ where $T_s^2 = (q-1)T_s + q$, $T_s x = yT_s + (q-1)x$.

The Hecke algebra

It can be deduced by Bernstein's presentation that there exist intertwining elements I_w in an extension of H such that

$$I_w \pi = w(\pi) I_w \tag{3}$$

$$I_s^2 = \frac{(1 - q\pi^{-a^\vee})(1 - q\pi^{a^\vee})}{(1 - \pi^{-a^\vee})(1 - \pi^{a^\vee})} = \frac{e_a e_{-a}}{d_a d_{-a}} = c_a c_{-a} \tag{4}$$

Corollary (Satake isomorphism)

The center of H is

$$Z(H) \cong R^W.$$

Lafforgue variety

Lafforgue variety

We will work in the more general context of a possibly non-commutative algebra H that is a finite free module over a commutative regular $A \subseteq Z(H)$.

Theorem

There is an open dense subscheme of an affine scheme $\mathrm{iLaf}_{H/A} \hookrightarrow \mathrm{Laf}_{H/A} = \mathrm{Spec} T_{H/A}$ parametrizing simple modules of H. It comes equipped with a finite projection $\mathrm{Laf}_{H/A} \to \mathrm{Spec} A$.

We call $T_{H/A}$ the *ring of traces* and it is the algebra generated by functions $f_h : \mathrm{iLaf}_{H/A} \to \mathbb{C}$ given by $f_h(V) = tr_V(h)$. There is a natural embedding $\mathrm{iLaf}_{H/A} \hookrightarrow \mathrm{Spec} T_{H/A}$ given by the functor of points formalism. The projection sends a simple module to its central character by Schur's lemma.

Lafforgue variety for GL_2

For GL_2, the irreducible Iwahori representations (discrete series) are of three kinds.

- Induced from a generic unramified character
- Characters
- Steinberg representations.

The first one is the largest component in the diagram and the last two categories are subquotients of the first one for non-generic characters.

In the last part of the talk we will see how to compute the ramification locus, ie. the dashed curve.

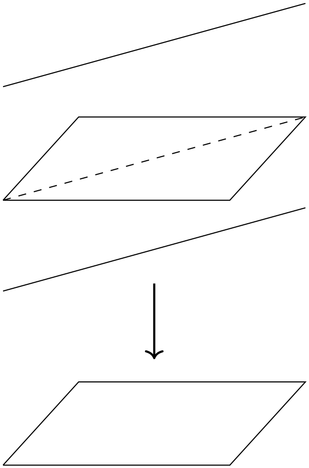

Figure: Projection from the Lafforgue variety to the Bernstein variety

Sketch of proof

We consider the following parametrizing spaces of isomorphism classes where all modules are over $H \otimes_A B$ and are flat as B-modules.

- $Hilb_{H/A}(B) = \{H \otimes_A B \twoheadrightarrow M\}$,
- $nHilb_{H/A}(B) = \{H \otimes_A B \twoheadrightarrow M \twoheadrightarrow N,\ rk(M) > rk(N)\}$,
- $V_{H/A}(B) = Hom_A(H, B)$,
- $G_{H/A}(B) = (H \otimes_A B)^{\times}$.

The first two functors are representable by proper schemes as closed subfunctors of representable functors. $V_{H/A} \cong \mathrm{Spec}(Sym_A(H))$ is affine and $G_{H/A}$ is a group scheme acting on all three spaces.

We have maps $nHilb_{H/A} \to Hilb_{H/A}$ given by forgetting N and $tr : Hilb_{H/A} \to V_{H/A}$ assigning to a module M the trace function given by M on $Sym_A(H)$. Both are $G_{H/A}$-equivariant.

Sketch of proof

This gives the sequence of $G_{H/A}$-equivariant maps between schemes over $\mathrm{Spec} A$

$$\begin{array}{ccccc} nHilb_{H/A} & \xrightarrow{F_N} & Hilb_{H/A} & \xrightarrow{tr} & V_{H/A} \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Spec}(A) & & \end{array}$$

We set $\mathrm{Laf}_{H/A} = im(tr)$, and $\mathrm{iLaf}_{H/A} = \mathrm{Laf}_{H/A} \setminus im(tr \circ F_N)$. Since $Hilb_{H/A}$ is proper, its image is proper and since $V_{H/A}$ is affine, we get $Laf_{H/A}$ is proper and affine thus finite over $\mathrm{Spec}(A)$.

By definition, $iLaf_{H/A}$ will correspond to simple modules, and since $nHilb_{H/A}$ is also proper $iLaf_{H/A}$ is an open subvariety, giving the result.

Trace Form and Discriminant

Trace Form

We will now try to detect the locus of reducibility. If M is a simple H-module, we recall the projection sends a module to its central character χ. Then M is an $H_\chi = H \otimes_{A,\chi} \mathbb{C}$-module.

We can thus consider simple H-modules by looking at the fibers over points of $\mathrm{Spec} A$. Then H_χ is a finite dimensional k-algebra whose semisimplification can be written $H_\chi / J(H_\chi) \cong \prod_{i=1}^{n_\chi} M_{k_i}(D_i)$ and n_χ is the number of irreducible representations.

Let $Tr_{H/A} : H \otimes_A H \to A$ be the bilinear map $Tr_{H/A}(h_1, h_2) = tr_{H/A}(h_1 h_2)$. Equivalently, we can think of $Tr_{H/A}$ as a map $Tr_{H/A} : H \to H^\vee := Hom(H, A)$. Then, $J(H_\chi) = \ker(Tr_{H/A,\chi})$.

Discriminant

Suppose n_χ is generically 1 and $J(H_\chi)$ generically trivial. After a non-canonical identification $H^\vee \cong H$, we can take the norm (determinant) of $Tr_{H/A}$, to get an element $d_{H/A} \in A$ well-defined up to $A^\times$. All these choices generate a principal ideal which we call the *discriminant* $d_{H/A}$ of H over A in analogy with the number ring case. Its zero set is the reducibility locus.

Lemma

Let $C/B/A$ be a tower of algebras such that A, B are commutative and regular, each extension is a finite locally free module over the previous one, and C is commutative. Then

$$d_{C/A} = d_{B/A}^{[C:B]} \cdot N_{B/A}(d_{C/B})$$

This follows from the exact sequence of Kahler differentials

$$0 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

Computation for GL_2

For $GL_2(F)$ we can consider the basis of H over $\mathbb{C}[x^{\pm}, y^{\pm}]$ given by $1, x, I_s, xI_s$. Then, keeping in mind $d_a = 1 - \pi^{-a^\vee} = 1 - xy^{-1} = y^{-1}(y - x)$,

$$Tr = \begin{pmatrix} 2 & x+y & 0 & 0 \\ x+y & x^2+y^2 & 0 & 0 \\ 0 & 0 & 2c_a c_{-a} & c_a c_{-a}(x+y) \\ 0 & 0 & c_a c_{-a}(x+y) & c_a c_{-a}(x^2+y^2) \end{pmatrix}, det(Tr) = e_a^2 e_{-a}^2$$

Notice the block-diagonal form of the trace form in this basis. This generalizes to a Zariski-local proof of the previous lemma. We retrieve for adjoint groups [3].

Theorem (Discriminant of adjoint groups)

For G adjoint, we have

$$d_{H/R^W} = \prod_{a \in \Phi} (e_a e_{-a})^{|W|^2/2}.$$

References I

- L. Lafforgue, “Le principe de fonctorialité de Langlands comme un probléme de généralisation de la loi d’addition,” 2016.
- J. Bernstein, R. Bezrukavnikov, and D. Kazhdan, “Deligne-lusztig duality and wonderful compactification,” 2017.
- S. Kato, “Irreducibility of principal series representations for Hecke algebras of affine type,” 1982.

Acknowledgment

Thank you all for listening!